GEOMETRY OF CR-SUBMANIFOLDS

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Dedicated

to

Islamic Philosophy and Way of Life

of which

My Parents are an Integral part

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Certificate

*This is to certify that this dissertation entitled " **Geometry of CR-Submanifolds"** has been written by **Miss. Tanveer Fatima** under my supervision as a partial fulfillment for the award of the degree of Master of Philosophy in Mathematics. She has fulfilled the prescribed conditions given in the statutes and ordinances of Aligarh Muslim University, Aligarh .*

I further certify that no part of this work in the present form has been submitted to any other university or institution for the award of any degree.

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PREFACE

The theory of submanifolds of a Kaehler manifold (or for that matter of any almost complex manifold with a Hermitian metric) presents an interesting geometry as its almost complex structure transforms a vector into a vector perpendicular to it, which naturally give rise to three special classes of submanifolds viz. holomorphic or invariant submanifolds (also known as almost complex submanifolds), totally real or anti-invariant submanifolds and slant submanifolds. In the first case the tangent space of submanifolds remain invariant under the action of J and in the second case it goes to the normal space under the action of J.

In 1978 A.Bejancu introduced the notion of CR-submanifolds of a Kaehler manifold which generalizes holomorphic as well as totally real submanifolds in the sense that they become particular cases of CR-submanifolds. A real submanifold M of an almost Hermitian manifold is called CR-submanifold if there exists a differentiable distribution $\mathcal{D}$ on M satisfying (i) $JD_x = D_x$ and (ii) $J(D_x^\perp) \subseteq T_x^\perp(M)$ for each $x \in M$, where $\mathcal{D}^\perp$ is the complimentary orthogonal distribution to $\mathcal{D}$.

CR-submanifolds are an active area of research for the past thirty years and play an important role in many diverse areas of differential geometry, relativity as well as in the mechanics [3], [12]. Integrability of the distributions give rise to the notion of CR-product submanifolds which are those CR-submanifolds that are totally Riemannian product of leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$. A lot of research has been done on CR-product submanifolds and characterizations are found for a CR-submanifold to become a CR-product submanifold (cf. [14], [18], [24]). Moreover, it is proved that there do not exist non trivial CR-products in hyperbolic spaces [46]. It was also found that S^6 does not admit non trivial CR-product submanifolds [46].

Bishop and O'Niell [6] in 1969 introduced warped product manifold as a generalization to Riemannian product manifolds. Easiest examples of warped product manifolds are surfaces of revolution. The study of warped products got impetus when B.Y.Chen studied warped product CR-submanifolds of a Kaehler manifold [15], [16]. After the impulse given by B.Y.Chen [15], [16], the study of warped product CR-submanifolds in Kaehler manifolds was extensively done only since 2001.

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Singly warped products or simply warped products were first introduced by Bishop and O'Niell [6] in 1969, in order to construct Riemannian manifolds with negative sectional curvature. In general, doubly warped products can be considered as generalization of singly warped products.

The warped and doubly warped product submanifolds form the main theme of this dissertation.

The dissertation comprises of four chapters and each chapter is divided into various sections. The mathematical relations obtained in the text have been labeled with double decimal numbering. The first figure denotes the chapter number, second represents the sections and the third point out the number of the definition, remark, equation, proposition, corollary or the theorem, as the case may be. For example, Theorem 1.2.3 refers to third theorem of second section in the first chapter.

The first chapter is introductory and contains those definition and results which are needed for the subsequent chapters. Moreover, this serves the purpose of making the dissertation as the terminology for the forthcoming chapters.

Chapter 2 deals with the warped product CR-submanifolds of Kaehler and nearly Kaehler manifolds. The warped product CR-submanifolds can be defined in two ways (i) $N_T \times_f N_\perp$ and (ii) $N_\perp \times_f N_T$. In the first case, it was found that they are not different from the CR-product submanifolds [15]. We have a general inequality in CR-warped product submanifolds of Kaehler manifolds and we also discuss an example for the existence of CR-warped product submanifolds $N_T \times_f N_\perp$ in nearly Kaehler manifolds. In the last section, we discuss the generalization of results of B.Y.Chen [15], [16] and Sahin [47] given in previous sections of this chapter. Further K.A.Khan, Shahid Ali and Nargis have extended this study to generic warped product submanifolds in Keahler manifolds.

In chapter 3 we pay our attention to warped product CR-submanifolds, CR-warped product submanifolds and doubly warped product CR-submanifolds in locally conformal Kaehler manifolds. We study a general inequality for CR-warped product submanfifolds in an l.c.K. manifold, we see that some anti-holomorphic CR-warped product submanifolds satisfying a certain condition in an l.c.K. manifold, satisfy the equlity and in proper CR-warped product submanifolds, its holomorphic submanifold in an l.c.K. space form is also an l.c.K. space form and its totally real submanifold is a real space form. In this chapter, we discuss a lot of essential and interesting properties of these submanifolds.

The last chapter deals with the analogue of warped and doubly warped product contact CR-submanifolds in trans-Sasakian manifolds and also we study semi-slant submanifolds in trans-Sasakian manifolds. First we discuss the integrability conditions of the distributions on semi-slant submanifolds and study the geometry of the leaves of these submanifolds, where it was found that $\mathcal{D} \oplus \mathcal{D}^{\perp}$ is never integrable. We also discuss the existence of warped product contact CR-submanifolds and the non-existence of doubly warped product contact CR-submanifolds in trans-Sasakian manifolds. We have discussed an example in this chapter showing that the warped product contact CR-submanifold of Kenmotsu manifold do exist.

In the end we have given a bibliography which by no mean is exhaustive, but contains only those references which are referred in the text.

CHAPTER 1
PRELIMINARIES

1.1. Introduction

The main objective of this introductory chapter is to discuss some basic concepts, preliminary notions and some fundamental results that are required for the development of the subject in the dissertation. We also discuss manifolds admitting different structures, space forms, theory of submanifolds, invariant and anti invariant submanifolds of complex and almost complex manifolds briefly. Although most of these results are readily available in research article and some in standard books e.g., Nomizu and Kobayshi [25], D.E.Blair [9], Yano [48], B.Y.Chen [12], A.Bejancu [3], we have collected them here for ready references and to set up our terminology.

1.2. Structures on C^∞-Manifolds

We can explain the geometry of a differentiable manifold by knowing a Riemannian metric on it. Further refined informations can be had by knowing additional structures on the manifold, for example, almost complex, almost Kaehler, nearly Kaehler and almost contact structures etc., ([21], [22]). In this section, we briefly discuss some of these structures.

By a Riemannian metric g on a manifold M, we mean a map $g : p \longrightarrow g_p$, where g_p is a positive definite inner product on $T_p(M)$. We require this map to be smooth in the sense that the function

$$p \longrightarrow g_{ij}(p) = g_p \left(\frac{\partial}{\partial x_i} |_p, \frac{\partial}{\partial x_j} |_p \right)$$

is smooth for all i, j on any chart (U, x) on M. On a paracompact manifold there exists a smooth Riemannian metric g.

For a Riemannian manifold (M, g) with a unique connection ∇, the Koszul formula is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \quad (1.2.1)$$

In what follows, we shall take a differentiable manifold which is connected and paracompact, so that it can always be endowed with a Riemannian metric g and a

Riemannian connection ∇

An almost complex structure on a real differentiable manifold $\bar{M}$ is a (1,1)-tensor field J which is at every point $p \in M$, an endomorphism of the tangent space $T_p(M)$ such that $J^2 = -I$, where I is the identity transformation. A manifold with a fixed almost complex structure is called an almost complex manifold. On an almost complex manifold, there always exists a Riemannian metric g invariant by the almost complex structure J i e , satisfying

$$g(JU, JV) = g(U, V) \tag{1.2 2}$$

for all $U, V \in T(\bar{M})$, where $T(\bar{M})$ is the tangent bundle of $\bar{M}$ By virtue of (1 2 2), g is called a Hermitian metric An almost complex manifold (resp a complex manifold) equipped with a Hermitian metric is called an almost Hermitian manifold (resp a Hermitian manifold)

Analogous to the almost complex structure J, there is defined a fundamental 2-form which plays an important role in the geometry as well as in the mechanics on the manifold [32] We describe it as follows

Let Φ denote the fundamental 2-form associated with the Hermitian metric g on $\bar{M}$ i e ,

$$\Phi(U, V) = g(U, JV) \tag{1 2 3}$$

for all vectors U, V in $T(\bar{M})$ Since g is invariant under J so is Φ i e ,

$$\Phi(JU, JV) = \Phi(U, V) \tag{1 2 4}$$

Definition 1.2.1. A symplectic form of dimension n on a real vector space V is a non degenerate exterior 2-form Φ of rank n If V admits a symplectic form Φ, then we say that Φ defines a symplectic structure on V or that (V, Φ) is a symplectic vector space

A symplectic structure on a manifold $\bar{M}$ is defined by a choice of a differentiable 2-form Φ satisfying following two conditions

(1) For all $p \in M$, Φ_p is non degenerate,

(2) Φ is closed i e , $d\Phi=0$

The almost complex structure J is not parallel in general with respect to the Riemannian connection $\bar{\nabla}$ on $\bar{M}$, defined by the Hermitian metric g In fact, we have the following formula

$$4g((\bar{\nabla}_U J)V, W) = 6d\Phi(U, JV, JW) - 6d\Phi(U, V, W) + g(N(V, W), JU), \tag{1 2 5}$$

where N is the Nijenhuis tensor of J, defined by

$$N(U,V) = 2([JU,JV] - [U,V] - J[U,JV] - J[JU,V]). \tag{1.2.6}$$

It is easy to verify that N satisfies

$$N(JU,V) = N(U,JV) = -JN(U,V). \tag{1.2.7}$$

It is well known that vanishing of the tensor $N(U,V)$ is the necessary and sufficient condition for an almost complex manifold to be a complex manifold [22].

If we extend the Riemannian connection $\bar{\nabla}$ to be a derivative on the tensor algebra of $\bar{M}$, then we have the following formulae

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V, \tag{1.2.8}$$

$$(\bar{\nabla}_U \Phi)(V,W) = g((\bar{\nabla}_U J), W). \tag{1.2.9}$$

We define a Kaehler manifold by using the fundamental 2-form Φ, almost complex structure J and the Riemannian metric g as follows:

Definition 1.2.2. A Hermitian metric on an almost complex manifold is called a Kaehler metric if the fundamental 2-form Φ is closed. A complex manifold equipped with a Kaehler metric is called a Kaehler manifold. In other words, an almost complex manifold $\bar{M}$ is Kaehler if

$$(\bar{\nabla}_U J)V = 0 \tag{1.2.10}$$

or equivalently,

$$\bar{\nabla}_U JV = J\bar{\nabla}_U V$$

for all U, V in $T(\bar{M})$. The connection $\bar{\nabla}$ on $\bar{M}$ is said to be a Kaehler connection.

A Hermitian manifold $\bar{M}$ is said to be nearly Kaehler if

$$(\bar{\nabla}_U J)V + (\bar{\nabla}_V J)U = 0$$

for all U, V in $T(\bar{M})$ and is almost Kaehler if

$$(\bar{\nabla}_U J)U + (\bar{\nabla}_{JU} J)JU = 0$$

for all U in $T(\bar{M})$.

For the relation among these classes, let us denote by K, AK, NK and H the classes of Kaehler, almost Kaehler, nearly Kaehler and Hermitian manifolds respectively. Then it can be easily seen that

$$\begin{array}{ccccc} K \subsetneq AK & & & & \\ \cap\!\!| & \text{and} & K \subseteq H, & & K = AK \cap NK. \\ NK & & & & \end{array}$$

Figure (1.2.1)

Remark 1.2.1. It is clear that every Kaehler manifold is nearly Kaehler, but converse need not be true in general.

For the Nijenhuis tensor N of J in nearly Kaehler manifold, we have

Proposition 1.2.1. Let $\bar{M}$ be a nearly Kaehler manifold. Then the Nijenhuis tensor N of J is given by

$$N(U, V) = 4J(\bar{\nabla}_V J)U$$

for any U, V in $T(\bar{M})$.

Definition 1.2.3. A Kaehler manifold $\bar{M}$ of constant holomorphic sectional curvature is called a complex space form. It is denoted by $\bar{M}(c)$.

The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(U, V)W = \frac{c}{4}\{g(V, W)U - g(U, W)V + g(JV, W)JU - g(JU, W)JV + 2g(U, JV)JW\} \qquad (1.2.11)$$

for any vector fields U, V, W tangent to $\bar{M}$.

We now discuss some examples of Kaehler and nearly Kaehler manifolds.

Example 1.2.1. Consider the complex n-space C^n with the metric

$$ds^2 = \sum_{j=1}^{n} dz^j d\bar{z}^j.$$

The fundamental 2-form Φ in this case is given by

$$\Phi = -i \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

Clearly, Φ is closed and so the metric defines a Kaehlerian structure on C^n. Thus, C^n is a complete, flat Kaehlerian manifold.

Example 1.2.2. Let CP^n be the complex projective space with homogeneous coordinates $z^0, z^1, z^2, ...z^n$. The complex quadratic Q^{n-1} is a complex hypersurface of CP^n defined by the equation

$$(z^0)^2 + (z^1)^2 + ...(z^n)^2 = 0.$$

Then, Q^{n-1} is a Kaehlerian manifold.

Example 1.2.3. S^6 with usual almost complex structure is nearly Kaehler but not Kaehler [46].

Let $\bar{M}$ be a $(2n+1)$-dimensional (i.e., odd dimensional) differentiable manifold. A triplet (ϕ, ξ, η) is said to be an almost contact structure on $\bar{M}$, where ϕ is a (1,1)-tensor field on $\bar{M}$, ξ is a vector field on $\bar{M}$ and η is a 1-form such that

$$\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{1.2.12}$$

where I is the identity map on $\bar{M}$. The vector field ξ is known as the structure vector field and 1-form η is the dual of ξ. A manifold $\bar{M}$ equipped with the almost contact structure (ϕ, ξ, η) is said to be an almost contact manifold. The condition (1.2.12) imply that

$$\phi\xi = 0 \quad \text{and} \quad \eta \circ \phi = 0. \tag{1.2.13}$$

Now suppose, there is given a Riemannian metric tensor field g on $\bar{M}$ which satisfies

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V) \tag{1.2.14}$$

for any vector fields U, V in $T(\bar{M})$. Then the structure (ϕ, ξ, η, g) is said to be an almost contact metric structure on $\bar{M}$. In this case, it is easy to check that

$$g(U, \xi) = \eta(U) \tag{1.2.15}$$

for any vector field U in $T(\bar{M})$.

An almost contact metric structure is called a contact metric structure if

$$d\eta = \Phi,$$

where Φ is a fundamental 2-form defined by

$$\Phi(U, V) = g(U, \phi V).$$

In this case for any vector field U in $T(\bar{M})$, we have

$$\bar{\nabla}_U \xi = -\phi U - \phi h U, \tag{1.2.16}$$

where $h = \frac{1}{2}L_\xi\phi$, $L_\xi\phi$ being the Lie derivative of ϕ with respect to ξ. The operator h satisfies

$$g(hU, V) = g(U, hV) \quad , \quad \phi \circ h = -h \circ \phi. \tag{1.2.17}$$

By a Sasakian manifold, we mean a contact metric manifold which is normal i.e.,

$$S_\phi + 2d\eta \otimes \xi = 0, \tag{1.2.18}$$

where S_ϕ is the Nijenhuis tensor of ϕ.

1.3. Submanifolds

If an n-dimensional differentiable manifold M admits an immersion

$$f : M \hookrightarrow \bar{M}$$

into an m-dimensional differentiable manifold $\bar{M}$, then M is said to be a submanifold of $\bar{M}$. Obviously $n \leq m$.

If $\bar{M}$ is a Riemannian manifold with a Riemannian metric g, then M also admits a Riemannian metric induced from $\bar{M}$ which is denoted by the same symbol g. The immersion f is said to be an isometric immersion if the differentiable map

$$f_\star : T(M) \hookrightarrow T(\bar{M})$$

preserves the Riemannian metric i.e., for $U, V \in T(M)$,

$$g(f_\star U, f_\star V) = g(U, V). \tag{1.3.1}$$

For every point $p \in M$, the tangent space $T_{f(p)}(\bar{M})$ of $\bar{M}$ admits the following decomposition

$$T_{f(p)}(\bar{M}) = T_p(M) \oplus T_p^\perp(M),$$

where $T_p(M)$ is the tangent space of M at p and $T_p^\perp(M)$ is the orthogonal compliment of $T_p(M)$ in $T_{f(p)}(\bar{M})$ consisting of all vectors normal to M.

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^\perp$ on $T(M)$ and $T^\perp(M)$ respectively, governed by the Gauss and Wiengarten formulae, viz,

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{1.3.2}$$

$$\bar{\nabla}_U N = -A_N U + \nabla_U^\perp N \tag{1.3.3}$$

for any tangent vector fields U, V on M and $N \in T^\perp(M)$. h and A_N are called second fundamental form and shape operator respectively and are related by

$$g(h(U, V), N) = g(A_N U, V). \tag{1.3.4}$$

For the second fundamental form h, we define the covariant differentiation $\bar{\nabla} h$ with respect to the connection in $T(M) \oplus T^{\perp}(M)$ by

$$(\bar{\nabla}_U h)(V,W) = \nabla_U^{\perp} h(V,W) - h(\nabla_U V, W) - h(V, \nabla_U W) \tag{1.3.5}$$

for any vector fields U, V and W tangent to M.

Using Gauss and Wiengarten formulas, we obtain the following celebrated equations due to Gauss, Coddazi and Ricci [12].

$$\bar{R}(U,V;W,Z) = R(U,V,W,Z) + g(h(U,Z),h(V,W)) - g(h(U,W),h(V,Z)), \tag{1.3.6}$$

$$\bar{R}((U,V)W)^{\perp} = (\bar{\nabla}_U h)(V,W) - (\bar{\nabla}_V h)(U,W), \tag{1.3.7}$$

$$\bar{R}(U,V;N_1,N_2) = R^{\perp}(U,V,N_1,N_2) - g([A_{N_1}, A_{N_2}]U,V) \tag{1.3.8}$$

for any vector fields U, V, W, Z tangent to M and N_1, N_2 are vector fields normal to M. In (1.3.7), $(\bar{R}(U,V)W)^{\perp}$ denotes the normal component of $\bar{R}(U,V)W$ and $R^{\perp}$ is the curvature tensor of the normal connection $\mathcal{D}$.

1.4. Some Special Submanifolds

Looking into the Guass formula, we can easily classify the submanifolds, putting conditions on h as follows:

Definition 1.4.1 [12]. A submanifold M of a Riemannian manifold $\bar{M}$ is said to be a totally geodesic submanifold if the second fundamental form h is identically zero on M i.e., $h \equiv 0$.

Definition 1.4.2 [12]. A submanifold M of a Riemannian manifold $\bar{M}$ is said to be a totally umbilical submanifold of $\bar{M}$ if its second fundamental form h satisfies

$$h(U,V) = g(U,V)H,$$

where $H = \frac{1}{n}(trace\ of\ h)$ is called the mean curvature vector and the squared norm of second fundamental form h is defined as

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)). \tag{1.4.1}$$

Definition 1.4.3 [12]. A submanifold M is called minimal if the mean curvature vector vanishes identically i.e., H=0.

Remark 1.4.1. A minimal totally umbilical submanifold is totally geodesic submanifold.

Definition 1.4.4. The vector sub-bundle μ of the normal bundle $T^{\perp}(M)$ is said to be parallel (in the normal bundle) if

$$\nabla_U^{\perp} N \in \mu$$

for any $U \in T(M)$ and any local cross-section N in μ.

On an almost Hermitian manifold $\bar{M}$,

$$g(JU, JV) = g(U, V)$$

for any vector fields U, V in $T(\bar{M})$. In other words

$$g(JU, U) = 0$$

i.e., $JU \perp U$ for each tangent vector field U on M.

Hence, for a submanifold M of $\bar{M}$ if $U \in T_p(M)$, JU may or may not belong to $T_p(M)$. Thus, action of the almost complex structure J on the tangent vectors of the submanifold of the almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifolds. These are defined as follows:

Definition 1.4.5 [41]. A submanifold of M an almost Hermitian manifold $\bar{M}$ is said to be invariant (or holomorphic) if

$$J(T_p(M)) \subset T_p(M)$$

for all $p \in M$.

Definition 1.4.6 [48]. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be anti-invariant (or totally real) if

$$J(T_p(M)) \subseteq T_p^{\perp}(M)$$

for all $p \in M$.

Remark 1.4.2. M is a holomorphic submanifold of $\bar{M}$ if for any nonzero vector U tangent to M at any point $p \in M$, the angle between JU and the tangent space $T_p(M)$ is equal to zero, whereas M is totally real if and only if for any non zero tangent vector U in M at any point $p \in M$, the angle between JU and $T_p(M)$ is equal $\pi/2$.

In 1978, A.Bejancu ([1], [2]) considered a new class of submanifolds of an almost Hermitian manifold of which the above classes namely invariant and totally real submanifolds are particular cases and named this class of submanifolds as CR-submanifolds that is, a CR-submanifold provides a single setting to study the invarian- and anti-invariant submanifolds of an almost Hermitian manifold.

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g and M be a Riemannian submanifold immersed in $\bar{M}$. At each point $p \in M$, let $\mathcal{D}_p$ be the maximal holomorphic subspace of the tangent space $T_p(M)$ i.e.,

$$\mathcal{D}_p = T_p(M) \cap JT_p(M).$$

If the dimension of $\mathcal{D}_p$ is same for all $p \in M$, we get a holomorphic distribution $\mathcal{D}$ on M.

Definition 1.4.7. A Riemannian submanifold M is said to be a CR-submanifold of an almost Hermitian manifold $\bar{M}$ if there exists a holomorphic distribution $\mathcal{D}$ on M such that its orthogonal complimentary distribution $\mathcal{D}^\perp$ is totally real i.e.,

$$J\mathcal{D}^\perp \subseteq T^\perp(M)$$

for all $p \in M$.

Clearly every real hypersurface M of an almost Hermitian manifold is a CR-submanifold, if $\dim(M) > 1$.

Remark 1.4.3. It is clear from the above definition that the dimension of $\mathcal{D}$ is always even and $J\mathcal{D}^\perp$ is a sub-bundle of $T^\perp(M)$, the normal bundle splits as

$$T^\perp(M) = J\mathcal{D}^\perp \oplus \mu,$$

where μ is the compliment of $J\mathcal{D}^\perp$ in $T^\perp(M)$ and μ is invariant under J.

Note. Throughout the dissertation M denotes a submanifold of the ambient space $\bar{M}$, unless mentioned otherwise.

Definition 1.4.8. A CR-submanifold M is called anti-holomorphic submanifold if

$$J\mathcal{D}_p^\perp = T_p^\perp(M)$$

for all $p \in M$.

Definition 1.4.9. A CR-submanifold M is said to be proper if neither $\mathcal{D}$ nor $\mathcal{D}^\perp = 0$. Obviously if $\mathcal{D} = 0$, then M is totally real submanifold and if $\mathcal{D}^\perp$=0, then M is holomorphic submanifold.

Definition 1.4.10. A CR-submanifold M is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$.

From the above definition, it is obvious that on a CR-product submanifold, the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$ are totally geodesic in M and vice-versa.

We know that the leaves of a distribution $\mathcal{D}$ on a manifold M are totally geodesic in M if and only if

$$\nabla_X Y \in \mathcal{D}$$

for all $X, Y \in \mathcal{D}$.

Thus, in the setting of CR-submanifold of an almost Hermitian manifold, the leaves of $\mathcal{D}$ are totally geodesic in M if and only if

$$\nabla_X Y \in \mathcal{D} \tag{1.4.2}$$

for all $X, Y \in \mathcal{D}$ which is equivalent to the condition

$$\nabla_X W \in \mathcal{D}^\perp \tag{1.4.3}$$

for all $X \in \mathcal{D}$ and $W \in \mathcal{D}^\perp$.

Similarly, for the totally geodesicness of the leaves of $\mathcal{D}^\perp$, the conditions

$$\nabla_Z W \in \mathcal{D}^\perp, \tag{1.4.4}$$

$$\nabla_Z X \in \mathcal{D} \tag{1.4.5}$$

for all X in $\mathcal{D}$ and Z, W in $\mathcal{D}^\perp$, are equivalent.

Now, we dicuss the condition of the totally geodesicness of the leaves of $\mathcal{D}$.

Lemma 1.4.1 [14]. The leaves of the holomorphic distribution $\mathcal{D}$ on a CR-submanifold M of a Kaehler manifold $\bar{M}$ are totally geodesic in M if and only if

$$g(h(\mathcal{D}, \mathcal{D}), J\mathcal{D}^\perp) = 0. \tag{1.4.6}$$

For the integrability of the distribution $\mathcal{D}^\perp$, we need the following lemma

Lemma 1.4.2 [14]. Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X), \tag{1.4.7}$$

$$A_{JZ}W = A_{JW}Z \tag{1.4.8}$$

for all X in $\mathcal{D}$ and Z, W in $\mathcal{D}^{\perp}$.

From the definition 1.4.10, a CR-submanifold is a CR-product if and only if the leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are totally geodesic in M. Hence by combining (1.4.2) and (1.4.5), we conclude that a CR-submanifold of an almost Hermitian manifold is a CR-product if and only if

$$\nabla_U X \in \mathcal{D} \tag{1.4.9}$$

for all $U \in T(M)$.

Similarly, by combining (1.4.3) and (1.4.4), it is concluded that a CR-submanifold is a CR-product if and only if

$$\nabla_U Z \in \mathcal{D}^{\perp}. \tag{1.4.10}$$

Conditions (1.4.9) and (1.4.10) are equivalent because

$$g(\nabla_U X, Z) = 0 \quad \Leftrightarrow \quad g(X, \nabla_U Z) = 0.$$

Next we have,

Theorem 1.4.1 [14]. A CR-submanifold of a Kaehler manifold is a CR-product if and only if

$$A_{J\mathcal{D}^{\perp}}\mathcal{D} = 0. \tag{1.4.11}$$

The generalization of Riemannian products namely warped product is defined as follows:

Definition 1.4.11 [15]. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and f be a positive differentiable function on B. The warped product of B and F is the Riemannian manifold

$$B \times_f F = (B \times F, g),$$

where $g = g_B + f^2 g_F$.

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|,$$

where π_i $(i = 1, 2)$ are the canonical projections of $B \times F$ onto B and F respectively and the function f is known as the warping function.

Definition 1.4.12 [41]. A doubly warped product (M, g) is a product manifold of the form $M =_f B \times_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$, where $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are smooth maps and g_B, g_F are the metrics on the Riemannian manifolds B and F respectively. If either $b \equiv 1$ or $f \equiv 1$ but not both, then we get a (single) warped product. If both $b \equiv 1$ and $f \equiv 1$, then we have a product manifold. If neither b nor f is constant, then we have a non trivial doubly warped product.

If $X \in \chi(B)$ and $Z \in \chi(F)$, then the Levi-Civita connection is

$$\nabla_X Z = (Zlnf)X + X(lnb)Z. \tag{1.4.12}$$

Bishop and O'Niell [6] obtained the following lemma which provides some basic formulae on warped product manifolds.

Lemma 1.4.3 [6]. Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in T(B)$ and $V, W \in T(F)$, then

(*i*) $\nabla_X Y \in T(B)$,

(*ii*) $\nabla_X V = \nabla_V X = \left(\frac{\mathrm{Xf}}{\mathrm{f}}\right) V$,

(*iii*) $\mathrm{nor}(\nabla_V W) = -\left(\frac{\mathrm{g(V,W)}}{\mathrm{f}}\right) \nabla f$,

where $\mathrm{nor}(\nabla_V W)$ is the component of $\nabla_V W$ in $T(B)$ and ∇f is the gradient vector field of the warping function f and is defined as

$$g(\nabla f, U) = Uf \tag{1.4.13}$$

for all $U \in T(M)$.

From (*ii*) of above lemma, we can see that

$$\nabla_U V = \nabla_V U = (Ulnf)V \tag{1.4.14}$$

for any vector field U tangent to B and V tangent to F.

For any vector field U tangent to M, we can decompose JU as

$$JU = PU + FU, \tag{1.4.15}$$

where PU and FU are the tangential and normal components of JU respectively. Then P is an endomorphism of the tangent bundle $T(M)$ and F is the normal bundle valued 1-form on $T(M)$.

Similarly, for any vector field N normal to M, if we put

$$JN = tN + fN, \tag{1.4.16}$$

where tN and fN are the tangential and normal components of JN respectively, then f can be treated as an endomorphism of the normal bundle $T^{\perp}(M)$ and t, a tangent bundle valued 1-form on $T^{\perp}(M)$ with kernel as $J\mathcal{D}^{\perp}$ and μ respectively.

The covariant differentiation of the operators P, F, t and f are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{1.4.17}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{1.4.18}$$

$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla^{\perp}_U N, \tag{1.4.19}$$

$$(\bar{\nabla}_U f)N = \bar{\nabla}_U fN - f\nabla^{\perp}_U N. \tag{1.4.20}$$

However, on a submanifold M of an almost contact manifold $(\bar{M},\ \phi,\ \xi,\ \eta)$ for any $U \in T(M)$, we also denote the tangential and normal components of ϕU by PU and FU respectively. Similarly, the tangential and normal components of ϕN for $N \in T^{\perp}(M)$ are denoted by tN and fN respectively i.e., we write

$$\phi U = PU + FU \tag{1.4.21}$$

and

$$\phi N = tN + fN. \tag{1.4.22}$$

The covariant differentiation of the operators P, F, t and f are defined in the same manner as in equations (1.4.17) to (1.4.20).

K.A.Khan, V.A.Khan and S.I.Husain [24] considered the tensors $\mathcal{P}$ and $\mathcal{Q}$ to obtain integrability conditions of the canonical distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on a CR-submanifold of an almost Hermitian manifold. They also studied the geometrical properties of the leaves of the distributions using these tensors.

Let $\bar{M}$ be an almost Hermitian manifold and M be a CR-submanifold of $\bar{M}$. Then for any U, V in $T(M)$, we have

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V.$$

Making use of Gauss and Weingarten formulae and equations (1.4.15) and (1.4.16), the above equation takes the form

$$(\bar{\nabla}_U J)V = (\nabla_U P)V - A_{FV}U - th(U,V) + (\nabla_U F)V + h(U,PV) - fh(U,V).$$

Furthermore, for any $U, V \in T(M)$, let us decompose $(\bar{\nabla}_U J)V$ into tangential and normal parts as

$$(\bar{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{1.4.23}$$

Now comparing the tangential and normal parts of $(\bar{\nabla}_U J)V$ in the above equation

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{1.4.24}$$

$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + h(U, PV) - fh(U, V) \tag{1.4.25}$$

Similarly, for $N \in T^{\perp}(M)$ denoting by $\mathcal{P}_U N$ and $\mathcal{Q}_U N$ respectively, the tangential and normal parts of $(\bar{\nabla}_U J)$, we get

$$\mathcal{P}_U N = (\bar{\nabla}_U t)N + PA_N U - A_{fN}U, \tag{1.4.26}$$

$$\mathcal{Q}_U N = (\bar{\nabla}_U f)N + h(tN, U) + fA_N U. \tag{1.4.27}$$

The following properties of $\mathcal{P}$ and $\mathcal{Q}$ are used in our subsequent sections of different chapters,

(p_1) (i) $\mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W$, (ii) $\mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W$

(p_2) (i) $\mathcal{P}_U(V + W) = \mathcal{P}_U V + \mathcal{P}_U W$, (ii) $\mathcal{Q}_U(V + W) = \mathcal{Q}_U V + \mathcal{Q}_U W$.

(p_3) (i) $g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W)$, (ii) $g(\mathcal{Q}_U V, \xi) = -g(V, \mathcal{P}_U \xi)$

(p_4) $\mathcal{P}_U JV + \mathcal{Q}_U JV = -J(\mathcal{P}_U V + \mathcal{Q}_U V)$.

Definition 1.4.13 [13]. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a generic submanifold if the maximal holomorphic subspace

$$\mathcal{D}_x = T_x(M) \cap JT_x(M)$$

has a constant dimension for each $x \in M$ and it defines a differentiable distribution on M. In this case the tangent space $T_x(M)$ of M at each point $x \in M$ is decomposed as

$$T_x(M) = \mathcal{D}_x \oplus \mathcal{D}_x^{\perp}.$$

Here, $\mathcal{D}_x^{\perp}$ is the orthogonal compliment of the holomorphic subspace $\mathcal{D}_x$ and is not necessarily totally real as was in the case of CR-submanifold. For this reason generic submanifold is a generalized version of CR-submanifold. The distribution $\mathcal{D}^{\perp}$ on a generic submani old is known as purely real distribution.

Now in view of the Remark 1.4.2, we have a third important class of submanifold of an almost Hermitian manifold (In particular of a Kaehler manifold), called slant submanifolds.

A slant submanifold is defined as submanifold of $\bar{M}$ such that for any non zero vector $U \in T_p(M)$, the tangent angle $\theta(U)$ between JU and the tangent space $T_x(M)$ is constant (which is independent of the choice of the point $x \in M$ and choice of the tangent vector $U \in T_x(M)$). It is clear that holomorphic and totally real submanifolds are special classes of slant submanifolds. A slant submanifold is called proper if it is neither holomorphic nor totally real submanifold.

If M is a slant submanifold of an almost Hermitian manifold $\bar{M}$, we have (cf., [19])

$$P^2 = -cos^2(\theta)\ I, \tag{1.4.28}$$

where I is identity map and θ is the writinger angle of M in $\bar{M}$. Hence, we have

$$g(PU, PV) = cos^2\theta\ g(U, V), \tag{1.4.29}$$

$$g(FU, FV) = sin^2\theta\ g(U, V) \tag{1.4.30}$$

for all U, V tangent to M. A natural generalization of CR-submanifolds in terms of slant distribution was given by N.Papaghiuc [45]. These submanifolds are known as semi-slant submanifolds. He defined these submanifolds as follows:

Definition 1.4.14 [45]. A submanifold M of an almost Hermitian manifold is called a semi-slant submanifold if it is endowed with two orthogonal complimentary distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that D is holomorphic and $\mathcal{D}^\theta$ is slant i.e., the angle $\theta(X)$ between JX and $\mathcal{D}^\theta_x$ is constant for each $X \in \mathcal{D}^\theta_x$.

Hence, CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\pi/2$ and $\mathcal{D} = \{0\}$ respectively.

For $\pi/2$, the semi-slant submanifold is semi-invariant submanifold. On a semi-slant submanifold M, for any $X \in T(M)$, we may write

$$X = P_1X + P_2X + \eta(X)\xi, \tag{1.4.31}$$

where $P_1X \in \mathcal{D}$ and $P_2X \in \mathcal{D}^\theta$.

Applying ϕ, (1.4.31) in view of (1.4.21) yields

$$\phi X_1 = \phi P_1X + TP_2X + NP_2X. \tag{1.4.32}$$

The differential geometry of semi-invariant or contact CR-submanifolds as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold was initiated by A.Bejancu and N.Papaghiuc [4] followed by several geometers.

Throughout our dissertation, for a submanifold M of an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$, we assume that the structure vector field ξ is tangential to the submanifold M and therefore the tangent bundle $T(M)$ is decomposed as

$$T(M) = \mathcal{D} \oplus < \xi > \oplus \mathcal{D}^{\perp},$$

where $< \xi >$ is the one dimensional distribution on M spanned by structure vector field ξ.

Definition 1.4.15. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a contact CR-submanifold (or semi-invariant submanifold) if there exists a pair of orthogonal distributions $(\mathcal{D}, \mathcal{D}^{\perp})$ satisfying the conditions;

(1) $T(M) = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus < \xi >$,

(2) The distribution $\mathcal{D}$ is invariant by ϕ i.e., $\phi \mathcal{D}_x = \mathcal{D}_x, \ x \in M$,

(3) The distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $\phi \mathcal{D}_x^{\perp} \subset T_x^{\perp}(M), \ x \in M$.

It follows that normal bundle splits as

$$T^{\perp}(M) = \phi \mathcal{D}^{\perp} \oplus \mu, \tag{1.4.33}$$

where μ is invariant sub-bundle of $T^{\perp}(M)$. If $\mathcal{D} = \{0\}$ (resp. $\mathcal{D}^{\perp} = \{0\}$), then M is said to be an anti-invariant (resp. invariant) submanifold. We say that M is proper contact CR-submanifold (or semi-invariant submanifold) if it is neither invariant nor anti-invariant.

Remark 1.4.4. Let M be a semi-invariant submanifold of an almost contact metric manifold $\bar{M}$. Then we have

(1) For any $U \in T(M)$, $PU \in \mathcal{D}$ and $FU \in \phi \mathcal{D}^{\perp}$,

(2) For any $N \in T^{\perp}(M)$, $tN \in \mathcal{D}^{\perp}$ and $fN \in \mu$.

CHAPTER 2

WARPED PRODUCT CR-SUBMANIFOLDS IN KAEHLER AND NEARLY KAEHLER MANIFOLDS

2.1. Introduction

In 1969, Bishop and O'Neill introduced the notion of warped product manifolds [6]. These manifolds are generalization of Riemannian product manifolds and occurs naturally, e.g., surfaces of revolution are warped product manifolds. In fact, spheres and $R^n \setminus \{0\}$ are also isometric to warped product manifolds.

In this Chapter we study warped product CR-submanifolds of Kaehler manifolds. The notion of CR-warped product submanifolds of the type $N_\perp \times_f N_\top$ and $N_\top \times_f N_\perp$ of a Kaehler manifold $\bar{M}$, where $N_\top$ is an invariant and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$, were introduced by B.Y.Chen [15] and studied by several others ([23], [26]). We also study the generalization of the results due to B.Y.Chen by K.A.Khan, V.A.Khan in nearly Kaehler setting et.al.[26]. The warped product submanifolds have attracted the attention of many researchers and many results on these submanifolds have been obtained recently [27].

In the last section of this chapter we discuss the generalization of the results of B.Y.Chen [15], [16] and Sahin [47]. They have shown that there are no proper warped product submanifolds of the type $M = N \times_f N_\top$ and $M = N_\top \times_f N$, where $N_\top$ is an invariant and N is any real non anti-invariant submanifold of a Kaehler manifold. Further, K.A.Khan, Ali S. and Nargis et.al.[28] have extended this study to generic warped product submanifolds of Kaehler manifolds.

We now recall that if B and F are Riemannian manifolds and $f > 0$ be a smooth function on B, the warped product $M = B \times_f F$ is defined as the product manifold $B \times F$ with metric

$$g = \pi_1^\star(g_B) + (f \circ \pi_1)^2 \pi_2{}^\star(g_F),$$

where π_1 and π_2 are projections of $B \times F$ onto B and F and g_B and g_F are metrics on B and F respectively. B is called the base of $M = B \times_f F$ and F is the fiber. We shall express the geometry of M in terms of warping function and the geometries of B and F. In case of a semi-Riemannian product it is easy to see that the fibers $p \times F = \pi_1^{-1}(p)$ and the leaves $B \times q = \pi_2^{-1}(q)$ are semi-Riemannian submanifolds of M.

We now illustrate above definition by an example.

Example 2.1.1. In spherical coordinates the line element of $R^3 \setminus \{0\}$ is given by

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2).$$

Setting $r = 1$ gives the line element of the unit sphere S^2. Evidently, $R^3 \setminus \{0\}$ is diffeomorphic to $R^+ \times S^2$ under the natural map $(t,p) \leftrightarrow tp$. Thus, the formula for ds^2 shows that $R^3 \setminus \{0\}$ can be identified with the warped product $R^+ \times_r S^2$, where r is the distance from origin Hence, the leaves are the rays from the origin and the fibers are the spheres $S^2(r)$, r>0.

Thus, $R^3 \setminus \{0\}$ is naturally isometric to $R^+ \times_r S^n$.

2.2. Warped Product CR-Submanifolds and CR-Warped Product Submanifolds in Kaehler Manifolds

The product submanifolds of the form $M = N_\perp \times_f N_T$ in a Kaehler manifold $\bar{M}$, where $N_\perp$ is totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$, are known as warped product CR-submanifolds. First, we shall consider the warped product submanifolds of the form $M = N_\perp \times_f N_T$ of a Kaehler manifold $\bar{M}$ and we see that proper warped product CR-submanifolds of a Kaehler manifold are trivial. Later, we shall consider warped products of the type $N_T \times_f N_\perp$ and discuss a characterization for these kinds of CR-warped product submanifolds. B.Y.Chen [15] has shown in previous case that the warped product submanifolds are nothing but CR-product submanifolds. To prove this, we need the following:

Lemma 2.2.1 [6]. On a warped product manifold $M = N_\perp \times_f N_T$, we have

$$\nabla_X Z = \nabla_Z X = Z(lnf)X$$

for each $X \in T(N_T)$ and $Z \in T(N_\perp)$.

Proof. Consider the vector fields X, Y in $T(N_T)$ and Z in $T(N_\perp)$, we have

$$[X, Z] = 0.$$

Therefore,

$$\nabla_X Z = \nabla_Z X.$$

Now, using (1.2.1) and the fact that N_T and $N_\perp$ are orthogonal to each other, we get

$$\begin{aligned} 2g(\nabla_Z X, Y) &= Zg(X,Y) \\ &= Zf^2 g_{N_T}(X,Y) \\ &= 2f(Zf)g_{N_T}(X,Y). \end{aligned}$$

Because,

$$g(X,Y) = g_{N_\perp}(X,Y) + f^2 g_{N_T}(X,Y), \tag{2.2.1}$$

$$g(\nabla_Z X, Y) = f^2 g_{N_T}(Z(lnf)X, Y)$$

or,

$$g(\nabla_Z X, Y) = g((Zlnf)X, Y).$$

Since,

$$g_{N_\perp}(X,Y) = 0$$

$$g(\nabla_Z X - Z(lnf)X, Y) = 0. \tag{2.2.2}$$

On the other hand, since $M = N_\perp \times_f N_T$ is a warped product, $N_\perp$ is a totally geodesic submanifold of M. Thus we have

$$g(\nabla_Z X, W) = -g(X, \nabla_Z W) = 0. \tag{2.2.3}$$

Combining (2.2.2) and (2.2.3), we get

$$g(\nabla_Z X - Z(lnf)X, U) = 0 \tag{2.2.4}$$

for any vector field U tangent on M, which implies that

$$\nabla_X Z = \nabla_Z X = Z(lnf)X.$$

From the proof of above lemma, it is easy to see that

Corollary 2.2.1. Let $M = N_\perp \times_f N_T$ be a warped product manifold then

(*i*) $N_\perp$ is totally geodesic in M.

(*ii*) N_T is totally umbilical in M.

We now discuss the theorem given by B.Y.Chen [15]

Theorem 2.2.1 [15]. If $M = N_\perp \times_f N_T$ is a warped product CR-submanifold of a Kaehler manifold $\bar{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$, then M is a CR-product.

Proof. Let $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of a Kaehler manifold $\bar{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$. From the Corollary 2.2.1, we know that in this case, $N_\perp$ is a totally geodesic submanifold of M i.e.,

$$\nabla_Z W \in T(N_\perp)$$

for any vector fields $Z, W \in T(N_\perp)$.

Using these facts in (1.2.1), we get

$$g([Z, W], Y) = 0 \tag{2.2.5}$$

for any $Z,\ W \in T(N_\perp)$ and $Y \in T(N_T)$.

Moreover, if $Y = \xi^i \frac{\partial}{\partial y^i}$ and $Z = \eta^j \frac{\partial}{\partial z^j}$ then

$$[Y, Z] = \xi^i \left(\frac{\partial}{\partial y^i}\eta^j\right)\frac{\partial}{\partial z^j} - \eta^j \left(\frac{\partial}{\partial z^j}\xi^i\right)\frac{\partial}{\partial y^i}$$

which shows that $[Y, Z] = 0$ as the ξ^i and η^j are constant with respect to y^i and z^j respectively.

Similarly, we can obtain

$$[Y, W] = 0. \tag{2.2.6}$$

Using the above two relations in (1.2.1), we get

$$\begin{aligned}2g(\nabla_Z W, Y) &= -Yg(Z, W)\\ &= -Y[g_{N_\perp}(Z, W) + f^2 g_{N_T}(Z, W)]\\ &= -Yg_{N_\perp}(Z, W)\\ &= 0\end{aligned}$$

which implies that $\nabla_Z W \in T(N_\perp)$. Thus for any vector fields $Z,\ W$ on $N_\perp$ and X on N_T, we have

$$g(\nabla_Z W, X) = 0. \tag{2.2.7}$$

Since, the ambient space $\bar{M}$ is Kaehler, we have

$$\bar{\nabla}_Z JW = J\bar{\nabla}_Z W.$$

Using Gauss and Weingarten formulas, we have

$$-A_{JW}Z + \nabla^\perp_Z(JW) = J(\nabla_Z W) + Jh(Z, W). \tag{2.2.8}$$

Taking product with JX, we get

$$g(-A_{JW}Z, JX) + g(\nabla^\perp_Z(JW), JX) = g(J(\nabla_Z W), JX) + g(Jh(Z, W), JX),$$

$$g(A_{JW}Z, JX) = -g(\nabla_Z W, X). \tag{2.2.9}$$

By combining (2.2.7) and (2.2.9), we obtain

$$g(h(\mathcal{D}, \mathcal{D}^\perp), J\mathcal{D}^\perp) = 0. \tag{2.2.10}$$

On the other hand, from Lemma 2.2.1, we have

$$\nabla_X Z = \nabla_Z X = (Zlnf)X \tag{2.2.11}$$

for any vector fields X in $T(N_T)$ and Z in $T(N_\perp)$. Thus, if we denote by h^T and A^T the second fundamental form and the shape operator of N_T in M, then by using Gauss and Weingarten formulas, we have

$$\begin{aligned} g(h^T(X,Y),Z) &= g(A_Z^T X,Y) \\ &= -g(\nabla_X Z,Y) \\ &= -Z(lnf)g(X,Y) \end{aligned} \tag{2.2.12}$$

for any vector fields X, Y on N_T and Z on $T(N_\perp)$. Hence, we get

$$h^T(X,Y) = -\nabla(lnf)g(X,Y), \tag{2.2.13}$$

where $\nabla(lnf)$ is the gradient of (lnf). From equation (2.2.13), N_T is a totally umbilical submanifold of M.

Let $\hat{h}$ denote the second fundamental form of N_T in the ambient space $\bar{M}$. Then

$$\hat{h}(X,Y) = h^T(X,Y) + h(X,Y) \tag{2.2.14}$$

for any X, Y tangent to N_T.

By applying (2.2.13) and (2.2.14), we get

$$g(\hat{h}(X,X),Z) = -Z(lnf)g(X,X). \tag{2.2.15}$$

Since, N_T is a holomorphic submanifold of $\bar{M}$, we also have the following relation

$$\hat{h}(X,JY) = \hat{h}(JX,Y) = J\hat{h}(X,Y). \tag{2.2.16}$$

Hence, by combining (2.2.15) and (2.2.16), we obtain

$$g(\hat{h}(X,X),Z) = -g(\hat{h}(JX,JX),Z) = Z(lnf)g(X,X). \tag{2.2.17}$$

Equations (2.2.15) and (2.2.17) imply that $Z(lnf) = 0$. Therefore by (2.2.12) and (2.2.14), we get

$$g(\hat{h}(X,Y),Z) = g(h^T(X,Y),Z) \tag{2.2.18}$$

for any X, Y in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$

Hence, by (2.2.14), (2.2.16) and (2.2.18)

$$g(h(X,Y),JZ) = g(\hat{h}(X,Y),JZ) = -g(\hat{h}(X,JY),Z) = 0.$$

Therefore,

$$g(h(\mathcal{D},\mathcal{D}),J\mathcal{D}^\perp) = 0. \tag{2.2.19}$$

Hence, (2.2.10) and (2.2.19) imply that

$$A_{J\mathcal{D}^{\perp}}\mathcal{D} = 0.$$

Therefore, by applying Theorem 1.4.1, we conclude that $M = N_{\perp} \times_f N_{\top}$ is a CR-product.

Lemma 2.2.2 [15]. For a CR-warped product $M = N_{\top} \times_f N_{\perp}$ in any Kaehler manifold $\bar{M}$, we have

(i) $g(h(\mathcal{D},\mathcal{D}), J\mathcal{D}^{\perp}) = 0$,

(ii) $\nabla_X Z = \nabla_Z X = X(lnf)Z$,

(iii) $g(h(JX,Z), JW) = X(lnf)g(Z,W)$,

(iv) $\nabla^{\perp}_X(JZ) = J\nabla_X Z$, whenever $h(\mathcal{D},\mathcal{D}^{\perp}) \subset J\mathcal{D}^{\perp}$,

(v) $g(h(\mathcal{D},\mathcal{D}^{\perp}), J\mathcal{D}^{\perp}) = 0$ if and only if $M = N_{\top} \times_f N_{\perp}$ is a trivial CR-warped product in $\bar{M}$,

where X, Y are vector fields on $N_{\top}$ and Z, W on $N_{\perp}$.

The following result due to [15] gives a characterization for a CR-submanifold to become a CR-warped product submanifold of a Kaehler manifold.

Theorem 2.2.2 [15]. A proper CR-submanifold M of a Kaehler manifold $\bar{M}$ is locally a CR-warped product if and only if

$$A_{JZ}X = ((JX)\lambda)Z \tag{2.2.20}$$

for $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$ and for some function λ on M satisfying $W\lambda = 0$, where $W \in \mathcal{D}^{\perp}$.

Proof. If M is a CR-warped product $N_{\top} \times_f N_{\perp}$ in a Kaehler manifold $\bar{M}$, then statement (i) and (ii) of Lemma 2.2.2 imply that

$$A_{JZ}X = -((JX)\ lnf)Z$$

for each $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Since f is a function on $N_{\top}$, we also have

$$W(lnf) = 0$$

for all $W \in \mathcal{D}^{\perp}$.

Conversely, assume that M is a proper CR-submanifold of a Kaehler manifold $\bar{M}$ satisfying

$$A_{JZ}X = ((JX)\lambda)Z \tag{2.2.21}$$

for $X \in \mathcal{D}$, $Z \in \mathcal{D}^{\perp}$ and for some function λ with $W\lambda = 0$, $W \in \mathcal{D}^{\perp}$. Thus we have

$$g(h(\mathcal{D}, \mathcal{D}), J\mathcal{D}^{\perp}) = 0. \tag{2.2.22}$$

It follows from (2.2.20), that the holomorphic distribution $\mathcal{D}$ is integrable and its leaves are totally geodesic in M. Also

$$\begin{aligned} g(((J^2X)\lambda)Z, W) &= g(((-X)\lambda)Z, W) \\ &= g(A_{JZ}JX, W) \\ &= g(-\bar{\nabla}_{JX}JZ, W) \\ &= g(J\bar{\nabla}_{JX}Z, W). \end{aligned}$$

Therefore,

$$\begin{aligned} -X(\lambda)g(Z, W) &= g(\bar{\nabla}_{JX}Z, JW) \\ &= g(h(JX, Z), JW). \end{aligned} \tag{2.2.23}$$

On the other hand, from Lemma 2.2.1 and (2.2.23), we have

$$\begin{aligned} g(\nabla_Z X, W) &= -g(\nabla_Z W, X) \\ &= -g(JA_{JW}Z, X) \\ &= g(h(JX, Z), JW) \\ &= -X(\lambda)g(Z, W). \end{aligned} \tag{2.2.24}$$

If we denote by $\nabla_Z^{\perp}$, the connection and $h^{\perp}$ as the second fundamental form of $N_{\perp}$ in M, then

$$g(X, \nabla_Z W) = g(X, \nabla_Z{}^{\perp} W) + g(X, h^{\perp}(Z, W))$$

or,

$$g(X, \nabla_Z W) = g(X, h^{\perp}(Z, W)).$$

From (2.2.24),

$$\begin{aligned} g(X, h^{\perp}(Z, W)) &= -X(\lambda)g(Z, W) \\ &= -g(\nabla(\lambda), X)\ g(Z, W) \\ &= -g(X, g(Z, W)\nabla(\lambda)). \end{aligned} \tag{2.2.25}$$

On the other hand, since $h^{\perp}(Z,W)$ is a vector normal to $\mathcal{D}^{\perp}$ and tangential to M,

$$g(Z',h^{\perp}(Z,W))=0 \tag{2.2.26}$$

for any vector field Z' on $\mathcal{D}^{\perp}$.

Combining (2.2.25) and (2.2.26), we get

$$\begin{aligned} g(U,h^{\perp}(Z,W)-g(Z,W)\nabla(\lambda)) &= 0, \\ h^{\perp}(Z,W) &= -g(Z,W)\nabla(\lambda) \end{aligned} \tag{2.2.27}$$

for $X\in\mathcal{D}$ and $Z,W\in\mathcal{D}^{\perp}$.

Using Weingarten formula,

$$\nabla_W H = -A_H W + \nabla^{\perp}_W H$$

for any vector field W tangent on $N_{\perp}$ and H normal on $N_{\perp}$, we have

$$\begin{aligned} g(\nabla_H W,X) &= g(\nabla_W H,X) \\ &= g(-A_H W,X)+g(\nabla^{\perp}_W H,X) \\ -g(W,\nabla_H X) &= g(-A_H W,X)+g(\nabla^{\perp}_W H,X). \end{aligned} \tag{2.2.28}$$

Since, $\mathcal{D}$ is totally geodesic in M,

$$g(W,\nabla_H X)=0.$$

Hence,

$$g(\nabla^{\perp}_W H,X)=0. \tag{2.2.29}$$

Also,

$$g(\nabla^{\perp}_W H,Z)=0. \tag{2.2.30}$$

Since, Z is a vector field tangent on $N_{\perp}$ and $\nabla^{\perp}_W H$ normal on $N_{\perp}$, combining (2.2.29) and (2.2.30), we get

$$\begin{aligned} g(\nabla^{\perp}_W H,U) &= 0 \\ \nabla^{\perp}_W H &= 0. \end{aligned} \tag{2.2.31}$$

Since, the totally real distribution $\mathcal{D}^{\perp}$ of a CR-submanifold of a Kaehler manifold is always integrable, (2.2.27) and (2.2.21) imply that each leaf of $\mathcal{D}^{\perp}$ is an extrinsic sphere in M i.e., a totally umbilical submanifold with parallel mean curvature vector. Thus, M is locally warped product $N_{\top}\times_f N_{\perp}$ of a holomorphic submanifold and totally real submanifold $N_{\top}$ of M, where $N_{\top}$ is a leaf of $\mathcal{D}$, $N_{\perp}$ is a leaf of $\mathcal{D}_{\perp}$ and f is a certain warping function (cf.,[14]).

2.3. A General Inequality for CR-Warped Product Submanifolds in Kaehler Manifolds

In this section we shall discuss a general inequality for CR-warped product submanifolds in a Kaehler manifold proved by B.Y. Chen [15].

Definition 2.3.1. For a real hyperspace M of a Kaehler manifold $\bar{M}$ with a unit normal vector field ξ, the tangent vector field $J\xi$ on M is called a characteristic vector field of M.

Definition 2.3.2. A unit tangent vector V on M is called a principal vector if V is an eigen vector of the shape operator A_ξ, the corresponding eigen value is called the principal curvature at V.

We now state the following lemma given by B.Y.Chen [15]

Lemma 2.3.1. Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then we have

(i) $g(\nabla_U Z, X) = g(JA_{JZ}U, X)$,

(ii) $A_{JZ}W = A_{JW}Z$,

(iii) $A_{JN}X = -A_N JX$

for any vector field U tangent to M, $X, Y \in \mathcal{D}$; $Z, W \in \mathcal{D}^\perp$ and $N \in \nu$.

Now, we discuss the following result for CR-warped products in Kaehler manifolds.

Theorem 2.3.1 [15]. Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold in a Kaehler manifold $\bar{M}$. We have

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2 \geq 2p||\nabla(lnf)||^2, \tag{2.3.1}$$

where p is the dimension of $N_\perp$.

(ii) If the equality sign of (2.3.1) holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold in $\bar{M}$.

(iii) When M is anti-holomorphic and $p > 1$, the equality sign of (2.3.1) holds identically if and only if $N_\perp$ is a totally umbilical submanifold of $\bar{M}$.

(*iv*) If M is anti-holomorphic and $p = 1$, then the equality sign of (2.3.1) holds identically if and only if the characteristic vector field $J\xi$ of M is a principal vector field with zero as its principal curvature. (Note that in this case, M is a real hypersurface in $\bar{M}$.) Also, in this case, the equality sign in (2.3.1) holds identically if and only if M is a minimal hypersurface in $\bar{M}$.

Proof . From Lemma 2.2.1, we have

$$g(h(Z, JX), JZ) = X(lnf), \tag{2.3.2}$$

where Z is any unit vector in $\mathcal{D}^{\perp}$.

Applying (2.3.2), we get inequality (2.3.1) immediately.

For any vector fields X in $\mathcal{D}$ and Z, W in $\mathcal{D}^{\perp}$, Lemma 1.4.2 and (1.3.4) imply that

$$g(\nabla_W Z, X) = g(JA_{JZ}W, X) = -g(h(JX, W), JZ). \tag{2.3.3}$$

Hence, by using Lemma 2.2.1 and (2.3.3), we get

$$g(\nabla_W Z, X) = -(Xlnf)g(Z, W). \tag{2.3.4}$$

On the other hand, if we denote by $h^{\perp}$, the second fundamental form of $N_{\perp}$ in $M = N_{\top} \times_f N_{\perp}$ we get

$$g(h^{\perp}(Z, W), X) = g(\nabla_W Z, X). \tag{2.3.5}$$

Combining (2.3.4) and (2.3.5), we get

$$h^{\perp}(Z, W) = -g(Z, W)\nabla(lnf). \tag{2.3.6}$$

Now, assume that the equality case of (2.3.1) holds, then we obtain from (2.3.2) that

$$h(\mathcal{D}, \mathcal{D}) = 0, \qquad h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) = 0, \qquad h(\mathcal{D}, \mathcal{D}^{\perp}) \subset J\mathcal{D}^{\perp}. \tag{2.3.7}$$

Since, $N_{\top}$ is a totally geodesic submanifold in M, the first condition in (2.3.7) implies that $N_{\top}$ is totally geodesic in $\bar{M}$.

On the other hand, (2.3.6) shows that $N_{\perp}$ is totally umbilical in M. Now, the second condition in (2.3.7) implies that $N_{\perp}$ is also totally umbilical in $\bar{M}$. Moreover, from (2.3.7), we know that M is minimal in $\bar{M}$.

Let us assume that M is an anti-holomorphic CR-warped product in $\bar{M}$. Then from statement (*i*) of Lemma 2.2.2, we get

$$h(\mathcal{D}, \mathcal{D}) = 0. \tag{2.3.8}$$

If $N_\perp$ is totally umbilical in $\bar{M}$, then there exists a normal vector field $\bar{H}$ of $N_\perp$ in $\bar{M}$ such that the second fundamental form $\hat{h}$ of $N_\perp$ in $\bar{M}$ satisfies

$$\hat{h}(Z, W) = g(Z, W)\bar{H} \tag{2.3.9}$$

for Z, W tangent to $N_\perp$.

Since,

$$\hat{h}(Z, W) = h^\perp(Z, W) + h(Z, W).$$

(2.3.9) implies that there is a normal vector field η such that

$$h(Z, W) = g(Z, W)\eta. \tag{2.3.10}$$

Hence, for each unit vector $W \in \mathcal{D}^\perp$ and each unit vector Z in $\mathcal{D}^\perp$ perpendicular to W, we have, on using Lemma 2.3.1

$$\begin{aligned} g(\eta, JW) &= g(h(W, Z), JW) \\ &= g(h(Z, W), JZ) \\ &= g(Z, W)\ g(\eta, JZ) \\ &= 0. \end{aligned} \tag{2.3.11}$$

Since, M is assumed to be anti-holomorphic, (2.3.11) implies either $p = 1$ or

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \tag{2.3.12}$$

Hence, (2.3.2), (2.3.8) and (2.3.12) implies the equality case of (2.3.1) holds whenever $p > 1$.

When $p = 1$, M is a real hypersurface of $\bar{M}$. In this case, the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature if and only if (2.3.12) holds.

So, in this case we also have equality case of (2.3.1) if the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature. From the first condition in (2.3.7), we also know that condition (2.3.12) holds if and only if M is minimal in $\bar{M}$.

By applying statement (ii), the converse is easy to verify.

In the following section we shall discuss the recent results by V.A.Khan, K.A.Khan et.al.[26] which extend the study of warped product CR-submanifolds of Kaehler manifolds by B.Y.Chen [15] to the setting of nearly Kaehler manifolds.

2.4. Warped Product CR-Submanifolds and CR-Warped Product Submanifolds in Nearly Kaehler Manifolds

Throughout this section, we assume that $\bar{M}$ is a nearly Kaehler manifold and $M = N_\perp \times_f N_\top$ is a warped product CR-submanifold of $\bar{M}$ and then we shall study the warped product CR-submanifolds of the type $N_\top \times_f N_\perp$ in a nearly Kaehler manifold $\bar{M}$.

The nearly Kaehler structure on an almost Hermitian manifold $\bar{M}$ can be characterized by

$$(a)\ \ \mathcal{P}_U V + \mathcal{P}_V U = 0, \qquad\qquad (b)\ \ \mathcal{Q}_U V + \mathcal{Q}_V U = 0 \tag{2.4.1}$$

for each $U,\ V \in T(M)$, where P and Q denotes the tangential and normal components of ∇J.

On the submanifold M of $\bar{M}$, by property (p_4) of $\mathcal{P}$ and $\mathcal{Q}$ as mentioned in chapter 1, we also have

$$\mathcal{P}_X JX + \mathcal{Q}_X JX = 0 \tag{2.4.2}$$

for each $X \in T(N_\top)$.

By Corollary 2.2.1, $N_\perp$ is totally geodesic in M and $N_\top$ is totally umbilical in M. Thus, if $h^\top$ and $\hat{h}$ denote the second fundamental forms of the immersions of $N_\top$ in M and in $\bar{M}$ respectively, then

$$\hat{h}(X,Y) = h^\top(X,Y) + h(X,Y), \tag{2.4.3}$$

$$h^\top(X,Y) = -g(X,Y)\nabla(lnf) \tag{2.4.4}$$

for each $X,\ Y \in T(N_\top)$.

The Lemma 2.2.1 can be restated as

$$\nabla_X Z = \nabla_Z X = (Zlnf)X. \tag{2.4.5}$$

Hence,

$$g(\nabla_X Z, X) = (Zlnf)\|X\|^2 = g(\nabla_{JX} Z, JX). \tag{2.4.6}$$

Applying formulae (1.3.2), (2.4.1) and (2.4.2), equation (2.4.6) can be written as

$$(Zlnf)\|X\|^2 = g(JZ, h(X, JX)). \tag{2.4.7}$$

Replacing X by JX in equation (2.4.7), we get

$$Zlnf)\|X\|^2 = -g(JZ, h(X, JX)). \tag{2.4.8}$$

From equations (2.4.7) and (2.4.8),

$$(Zlnf)\|X\|^2 = 0. \tag{2.4.9}$$

If M is assumed to be a proper warped product CR-submanifold, then $Z(lnf) = 0$ i.e., M is simply a CR-product. In other words, the theorem of B.Y.Chen [15] is extended to the setting of nearly Kaehler manifold as

Theorem 2.4.1 [26]. There does not exist a proper warped product CR-submanifold $N_\perp \times_f N_T$ in nearly Kaehler manifolds.

Lemma 2.4.1. Let M be a warped product CR-submanifold of a nearly Kaehler manifold $\bar{M}$. Then we have

(*i*) $g(h(X,Y), JZ) = 0$

(*ii*) $g(\nabla_Z X, W) = X(lnf)g(Z,W) = g(h(JX,Z), JW)$,

for each $X, Y \in T(N_T)$ and $Z, W \in T(N_\perp)$.

Proof. By equations (1.4.24) and (1.4.25),

$$g(A_{FZ}X, Y) = g(\nabla_X Z, JY) - g(\mathcal{P}_X Z, Y).$$

The first term in the right hand side of the above equation is zero in view of Lemma 2.2.1.

Thus, equation reduces to

$$g(A_{FZ}X, Y) = -g(\mathcal{P}_X Z, Y).$$

The left hand side of the above equation is symmetric in X and Y whereas the right hand side is skew symmetric in X and Y . That proves

$$g(h(X,Y), JZ) = g(\mathcal{P}_X Z, Y) = 0.$$

The first equality in (*ii*) is an immediate consequence of Lemma 2.2.2 (*ii*).

For the second equality, by Gauss formula, we may write

$$\begin{aligned} g(h(JX,Z), JW) &= g(\bar{\nabla}_Z JX, JW) \\ &= g(Q_Z X, JW) + g(\nabla_Z X, W) \\ &= g(Q_Z JX, W) + X(lnf)g(Z,W) \\ &= -g(\mathcal{P}_Z W, JX) + X(lnf)g(Z,W). \end{aligned}$$

The first term in the right hand side of the above equation is zero by virtue of (2.2.16) and the equation reduces to

$$g(h(JX,Z),JW) = (Xlnf)g(Z,W)$$

which completes the proof of statement (ii).

Theorem 2.4.2 [26]. Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ with integrable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Then M is locally a CR-warped product if and only if

$$A_{JZ}X = -(JX\lambda)Z \tag{2.4.10}$$

for each $X \in \mathcal{D}, Z \in \mathcal{D}^{\perp}$ and λ, a C^{∞}-function on M such that $W\lambda = 0$ for each $W \in \mathcal{D}^{\perp}$.

Proof. If M is a warped product CR-submanifold $N_{\mathrm{T}} \times_f N_{\perp}$, then on applying Lemma 2.4.1, we obtain equation (2.4.10). In this case $\lambda = lnf$.

Conversely, suppose

$$A_{JZ}X = -(JX\lambda)Z,$$

then

$$g(h(X,Y),JZ) = 0$$

i.e., $h(X,Y) \in \lambda$ for each $X,\ Y \in \mathcal{D}$.

As $\mathcal{D}$ is assumed to be integrable, by (2.2.14), $Q_XY = 0$ and therefore by formula (1.4.25),

$$F\nabla_XY = h(X,JY) - fh(X,Y).$$

As $h(X,Y) \in \lambda$ for each $X,\ Y \in \mathcal{D}$, $FU \in J\mathcal{D}^{\perp}$ for each $U \in T(M)$ and $f\xi \in \mu$ for all $\xi \in T^{\perp}(M)$, we deduce from the above equation that $\nabla_XY \in \mathcal{D}$. That means, leaves of $\mathcal{D}$ are totally geodesic in M. Now,

$$\begin{aligned} g(\nabla_ZW,X) &= g(J\bar{\nabla}_ZW,JX) \\ &= -g(\mathcal{P}_ZW,JX) - g(A_{JW}Z,JX). \end{aligned}$$

The first term in the right hand side of the above equation vanishes in view of (2.2.16) and the second term on making use of (2.4.10) reduces to $X\lambda\ g(Z,W)$. That is, we have

$$g(\nabla_ZW,X) = X\lambda\ g(Z,W). \tag{2.4.11}$$

Now, by Gauss formula

$$g(h^{\perp}(Z,W),X) = g(\nabla_ZW,X),$$

where $h^{\perp}$ denotes the second fundamental form of the immersion of $N_{\perp}$ into M. Using (2.4.11), the last equation gives

$$g(h^{\perp}(Z,W),X) = X\lambda\, g(Z,W)$$

which shows that each leaf $N_{\perp}$ of $\mathcal{D}^{\perp}$ is totally umbilical in M. Moreover, the fact that $W\lambda = 0$ for all $W \in \mathcal{D}^{\perp}$, implies that the mean curvature vector on $N_{\perp}$ is parallel along $N_{\perp}$ i.e., each leaf of $\mathcal{D}^{\perp}$ is an extrinsic sphere in M. Thus, M is locally a warped product $N_{\top} \times_f N_{\perp}$ of a holomorphic submanifold $N_{\top}$ and a totally real submanifold $N_{\perp}$ of M [12]. Here $N_{\top}$ is a leaf of $\mathcal{D}$ and $N_{\perp}$ is a leaf of $\mathcal{D}^{\perp}$ and f is a warping function.

Now, we discuss an example showing that proper CR-warped product submanifolds $N_{\top} \times {}_f N_{\perp}$ do exist in nearly Kaehler manifolds. This example was initially given by K.Sekigawa [46].

EXAMPLE 2.4.1. Let $S^2 = \{y = (y_2, y_4, y_6) \in \mathbb{R}^3;\ y_2^2 + y_4^2 + y_6^2 = 1\}$ be a unit 2-sphere and $S^1 = \{z = e^{\sqrt{-1}\,t},\ t \in \mathbb{R}\}$ a unit circle. Let ψ be the C^∞-mapping from the product manifold $S^2 \times S^1$ into S^6 i.e.,

$$\psi : S^2 \times S^1 \longrightarrow S^6$$

defined as

$$\begin{aligned}\psi(y,z) &= \psi\big((y_2,y_4,y_6), e^{\sqrt{-1.t}}\big)\\ &= (y_2\cos t)e_2 - (y_2 \sin t)e_3 + (y_4\cos 2t)e_4\\ &\quad + (y_4 \sin 2t)e_5 + (y_6\cos t)e_6 - (y_6\sin t)e_7\end{aligned}$$

for $y = (y_2, y_4, y_6) \in S^2$ and $z = e^{\sqrt{-1}\,t} \in S^1,\ t \in \mathbb{R}$.

Then we may easily check that ψ gives rise to an isometric immersion from the warped product Riemannian manifold $S^2 \times {}_f S^1$ into S^6, where f is the warping function on S^2 which is given by the restriction of the function

$$F(y_2,y_4,y_6) = \sqrt{(1+3y_4^2)}$$

on $\mathbb{R}^3$ to S^2.

This is an example of 3-dimensional proper CR-warped product submanifolds of S^6 such that both the holomorphic distribution and totally real distribution are integrable.

2.5. Generic Warped Product Submanifolds in Kaehler Manifolds

In this section, we study generic warped product submanifolds of a Kaehler manifold M of the form $M = N_T \times_f N$, $M^f = N \times_f N_T$ respectively, where N_T is a holomorphic submanifold and N is any real non anti-invariant submanifold of M.

Now, we have

Theorem 2.5.1 [28]. There do not exists proper generic warped product submanifold $M = N \times_f N_T$ of a Kaehler manifold $\bar{M}$, where N_T is invariant submanifold and N is any real non anti-invariant submanifold of M.

Proof. For any vector field X in $T(N_T)$ and $U \in T(M)$ using (2.2.11), we get

$$\begin{aligned} g(\bar{\nabla}_X X, U) &= -g(\bar{\nabla}_X U, X) \\ &= -g(\nabla_X U, X) \\ &= -Ulnf \; \|X\|^2. \end{aligned} \tag{2.5.1}$$

Also, we have

$$\begin{aligned} g(\bar{\nabla}_X X, U) &= g(J\bar{\nabla}_X X, JU) \\ &= g(\bar{\nabla}_X JX, JU) \\ &= -g(\bar{\nabla}_X JU, JX) \\ &= -g(\bar{\nabla}_X PU, JX) - g(\bar{\nabla}_X FU, JX) \\ &= -PU \; lnf \; g(X, JX) + g(A_{FU}X, JX) \\ &= g(h(X, JX), FU). \end{aligned} \tag{2.5.2}$$

Thus from (2.5.1) and (2.5.2), we get

$$g(h(X, JX), FU) = -Ulnf \; \|X\|^2. \tag{2.5.3}$$

Now, replacing X by JX in (2.5.3), we get

$$\begin{aligned} g(h(JX, J^2X), FU) &= -Ulnf \; \|X\|^2 \\ -g(h(X, JX), FU) &= -Ulnf \; \|X\|^2 \\ g(h(X, JX), FU) &= Ulnf \; \|X\|^2. \end{aligned} \tag{2.5.4}$$

Thus from (2.5.3) and (2.5.4), we get

$$U\ln f\|X\|^2 = 0$$

for all $U \in T(M)$ which implies that f is constant or $X = 0$.

Hence, the theorem is proved.

If the factors N and N_{T} are interchanged in the above results, then we have

Theorem 2.5.2 [28]. There do not exists proper generic warped submanifold $M = N_{\mathrm{T}} \times_f N$ of a Kaehler manifold $\bar{M}$, where N_{T} is a holomorphic submanifold and N is any real non anti-invariant submanifold of M.

Proof. For any vector fields U, V in $T(M)$ and using the fact that M is Kaehler we have

$$\bar{\nabla}_U JV = J\bar{\nabla}_U V,$$

Therefore, on using (1.4.15) and (1.3.2), we get,

$$\nabla_U PV + \nabla_U FV = J(\nabla_U V + h(U,V)).$$

Again by using (1.3.2), (1.3.3),(1.4.15) and (1.4.16), we have

$$\nabla_U PV + h(U,PV) - A_{FV}U + \nabla_U^{\perp} FV = P\nabla_U V + F(\nabla_U V) + th(U,V) + fh(U,V).$$

Now, comparing tangential part and using (1.4.17), we obtain

$$(\bar{\nabla}_U P)V = A_{FV}U + th(U,V). \tag{2.5.5}$$

On using (2.2.11), for any vector field $X \in T(N_{\mathrm{T}})$, we get

$$\begin{aligned}
(\bar{\nabla}_X P)U &= \nabla_X PU - P\nabla_X U \\
&= (X\ln f)PU - (X\ln f)PU \\
&= 0.
\end{aligned}$$

Using it in (2.5.5),

$$A_{FU}X = -th(X,U). \tag{2.5.6}$$

On the other hand, we have

$$(\nabla_U P)X = (PX\ln f)U - (X\ln f)PU. \tag{2.5.7}$$

Also from (2.5.5), we have

$$(\nabla_U P)X = th(X,U). \tag{2.5.8}$$

Thus, on combining (2.5.7) and (2.5.8), we have

$$(PXlnf)U = (Xlnf)PU + th(X,U). \tag{2.5.9}$$

From (2.5.6) and (2.5.9), it follows that

$$(PX\ lnf)U - (X\ lnf)PU = -A_{FU}X.$$

Now, taking inner product with PU in above equation

$$g(h(X,PU),FU) = Xlnf\|PU\|^2. \tag{2.5.10}$$

Now, for any $U \in T(N)$ and $X \in T(N_T)$, we have

$$g(\bar{\nabla}_{PU}U,X) = 0, \tag{2.5.11}$$

On using the fact that $J\bar{\nabla}_{PU}U = \bar{\nabla}_{PU}JU$ in (2.5.11), we get

$$\begin{aligned}
0 &= g(\bar{\nabla}_{PU}JU,JX, \\
&= g(\bar{\nabla}_{PU}PU,JX) + g(\bar{\nabla}_{PU}FU,JX) \\
&= g(\bar{\nabla}_{PU}PU,JX) - g(A_{FU}PU,JX) \\
&= -g(\bar{\nabla}_{PU}JX,PU) - g(h(PU,JX),FU) \\
&= -JXlnf\|PU\|^2 - g(h(JX,PU),FU). \\
-g(h(JX,PU),FU) &= JXlnf\|PU\|^2. \qquad (2.5.12)
\end{aligned}$$

Replacing X by JX in (2.5.12), we get

$$-g(h(X,PU),FU) = Xlnf\|PU\|^2. \tag{2.5.13}$$

Now, (2.5.10) and (2.5.13) implies that

$$Xlnf = 0.$$

Thus, f is constant or $X = 0$, which proves the result.

CHAPTER 3

WARPED AND DOUBLY WARPED PRODUCT CR-SUBMANIFOLDS IN LOCALLY CONFORMAL KAEHLER MANIFOLDS

3.1. Introduction

Recently B.Y.Chen [15] introduced the notion of warped product CR-submanifolds and CR-warped products of Kaehler manifolds that is, a warped product Riemannian submanifold of a holomorphic submanifold and a totally real submanifold in a Kaehler manifold ([15]).

In this chapter we pay our attention to such submanifolds in locally conformal Kaehler (l.c.K.) manifolds. We study a general inequality for CR-warped product submanifolds in locally conformal Kaehler manifolds and we see that some anti-holomorphic CR-warped product submanifolds satisfying a certain condition in an l.c.K. manifold, satisfy the equality and in a proper CR-warped product which satisfies the equality, we see that its holomorphic submanifold in an l.c.K. space form is also an l.c.K. space form and its totally real submanifold is a real space form . We also discuss doubly warped product CR-submanifolds in locally conformal Kaehler manifolds. The result which we discuss in this chapter are mainly due to [8], [11], [38], [15] etc.

3.2. Some Basic Results

In this section we recall some basic definitions and results on locally conformal Kaehler manifolds which will be useful in subsequent sections of this chapter. The integrability conditions for the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on a CR-submanifold of a locally conformal Kaehler manifold $\bar{M}$ have been discussed. Also we discuss the parallelness of the endomorphism T.

A Hermitian Manifold $\bar{M}$ with structure (J, g) is called a locally conformal Kaehler (l.c.K.) manifold if each point $x \in \bar{M}$ has an open neighborhood U with a differentiable function $\rho : U \longrightarrow R$ such that $g^* = e^{-2\rho} g \mid_U$ is a Kaehlerian metric that is, $\nabla^* J = 0$, where J is the almost complex structure and $\nabla^* J$ is the covariant differentiation with respect to g^*.

Then we have

Proposition 3.2.1 [30]. A Hermitian manifold $\bar{M}$ with structure (J, g) is l.c.K. if and only if there exists a global 1-form α, called the Lee form, satisfying

$$d(\alpha) = 0, \tag{3.2.1}$$

$$(\bar{\nabla}_Y J)X = -g(\alpha^{\#}, X)JY + g(X,Y)\beta^{\#} + g(JX,Y)\alpha^{\#} - g(\beta^{\#}, X)Y \tag{3.2.2}$$

for any $X, Y \in \Gamma T(M)$, where $\bar{\nabla}$ denotes the covariant differentiation with respect to g, $\alpha^{\#}$ is the dual vector field of α, the 1-form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^{\#}$ is the dual vector field of β and $\Gamma T(\bar{M})$ means the set of all differentiable vector field on $\bar{M}$.

An l.c.K. manifold $\bar{M}$ is called an l.c.K. space form if it has a constant holomorphic sectional curvature c (we write it $\bar{M}(c)$) [30]. Then the Riemannian curvature tensor $\bar{R}$ of an l.c.K. space form $\bar{M}(c)$ with constant holomorphic sectional curvature c is given by

$$\begin{aligned}4\bar{R}(X,Y,Z,W) &= c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(JX,W)g(JY,Z) - g(JX,Z)\\ &g(JY,W) - 2g(JX,Y)g(JZ,W)\} + 3\{P(X,W)g(Y,Z) - P(X,Z)g(Y,W) + g(X,W)\\ &P(Y,Z) - g(X,Z)P(Y,W)\} - \tilde{P}(X,W)g(JY,Z) + \tilde{P}(X,Z)g(JY,W) - g(JX,W)\tilde{P}(Y,Z)\\ &+ g(JX,Z)\tilde{P}(Y,W) + 2\{\tilde{P}(X,Y)g(JZ,W) + g(JX,Y)\tilde{P}(Z,W)\}\end{aligned} \tag{3.2.3}$$

for any $X, Y, Z, W \in \Gamma T(\bar{M}(c))$, where P is (0,2)-symmetric tensor and we put

$$P(X,Y) = -(\bar{\nabla}_Y \alpha)X - \alpha(X)\alpha(y) + \frac{1}{2}\|\alpha\|^2 g(X,Y) \tag{3.2.4}$$

and

$$\tilde{P}(X,Y) = -P(X,JY). \tag{3.2.5}$$

Remark 3.2.1. To get (3.2.3), we have to assume that the symmetric (0,2)-tensor P is hybrid i.e.,

$$P(JX,Y) + P(X,JY) = 0$$

or equivalently, $\tilde{P}$ is skew-symmetric (0,2)-tensor [30].

Remark 3.2.2. If we assume that P is hybrid in a generalized Hopf manifold, then the manifold is Kaehlerian one. So, the Riemannian curvature tensor of an l.c.K. space form with parallel Lee form is not represented in tensor.

Now, we have

Proposition 3.2.2 [35]. On a CR-submanifold M of an l.c.K. manifold $\bar{M}$, we have (*i*) the distribution $\mathcal{D}^{\perp}$ is integrable,

(*ii*) the distribution $\mathcal{D}$ is integrable if and only if

$$g(h(X,JY)-h(Y,JX)-2g(JX,Y)\alpha^{\#},JZ)=0$$

for any $X,Y\in\mathcal{D}$ and $Z\in\mathcal{D}^{\perp}$.

For a CR-product in an l.c.K. manifold, we have

Proposition 3.2.3 [35]. A CR-submanifold in an l.c.K. manifold $\bar{M}$ is a CR-product if and only if the endomorphism T is parallel or equivalently the following equation is satisfied;

$$A_{JZ}X=-g(\beta^{\#},X)Z+g(\beta^{\#},Z)X-g(\alpha^{\#},Z)JX \tag{3.2.6}$$

for any $X\in\mathcal{D}$ and $Z\in\mathcal{D}^{\perp}$.

Remark 3.2.3. In a proper CR-product of an l.c.K. manifold the Lee vector field $\alpha^{\#}$ is normal to $\mathcal{D}^{\perp}$ [35].

On a CR-submanifold M of an l.c.K. manifold $\bar{M}$, we know that the following relations are known [35].

$$g(\nabla_U Z,X)=g(JA_{JZ}U,X)+g(\alpha^{\#},Z)g(U,X)+g(U,Z)g(\alpha^{\#},X)-g(\beta^{\#},Z)g(JU,X), \tag{3.2.7}$$

$$A_{JZ}W=A_{JW}Z+g(\beta^{\#},Z)W-g(\beta^{\#},W)Z \tag{3.2.8}$$

and

$$A_N JX+A_{JN}X=g(\beta^{\#},N)-g(\alpha^{\#},N)JX \tag{3.2.9}$$

for any $U\in\Gamma T(M)$, $X\in\mathcal{D}$; $Z,\ W\in\mathcal{D}^{\perp}$ and $N\in\nu$.

Let (B,g_B) and (F,g_F) be Riemannian manifolds and ∇^B and ∇^F be the Levi-Civita connections of the Riemannian metrics g_B and g_F respectively. The Levi-Civita connections ∇ of the doubly warped product M is expressed as [8]

$$\left.\begin{aligned}\nabla_X Y&=\nabla^B_X Y-\frac{f^2}{b^2}g_B(X,Y)\nabla^F(lnf),\\ \nabla_X Z&=Z(lnf)X+X(lnb)Z\end{aligned}\right\} \tag{3.2.10}$$

for all X,Y tangent to B and Z tangent to F. Here, $\nabla^F(lnf)$ denotes the gradient of lnf with respect to the metric g_F.

3.3. Warped Product CR-Submanifolds and CR-Warped Product Submanifolds in l.c.K. Manifolds

Let $N_{\top}$ (resp. $N_{\perp}$) be a holomorphic (resp. totally real) submanifold in an l.c.K. manifold $\bar{M}$. We consider a warped product submanifold of the form $M_1 = N_{\perp} \times_f N_{\top}$ with a warping function $f(>0) \in C^{\infty}(N_{\perp})$, where $C^{\infty}(N_{\perp})$ means the set of all differentiable functions on $N_{\perp}$. We call such a submanifold, a warped product CR-submanifold in an l.c.K. manifold $\bar{M}$.

Next, we consider a CR-warped product CR-submanifold $M_2 = N_{\top} \times_f N_{\perp}$ with a warping function $f(>0) \in C^{\infty}(N_{\top})$ in an l.c.K. manifold $\bar{M}$. We call such a submanifold, a CR-warped product. In a CR-warped product $M_2 = N_{\top} \times_f N_{\perp}$ in an l.c.K. manifold $\bar{M}$, it is easy to show that $N_{\top}$ is totally geodesic in M_2 that is , $\nabla_X Y \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$.

Now, we have a proposition:

Proposition 3.3.1 [11]. On a warped product CR-submanifold $M_1 = N_{\perp} \times_f N_{\top}$ of an l.c.K. manifold $\bar{M}$, the holomorphic submanifold $N_{\top}$ is totally umbilical in M.

Proof. It is easy to show that $N_{\perp}$ is totally geodesic in M_1

that is, $$\nabla_W Z \in \Gamma T(N_{\perp})$$

for any $Z, W \in \Gamma T(N_{\perp})$.

This means that

$$g(\nabla_W Z, X) = 0 \tag{3.3.1}$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$, where we put $\mathcal{D} = \Gamma T(N_{\top})$ and $\mathcal{D}^{\perp} = \Gamma T(N_{\perp})$.

Using (3.2.2) and (3.3.1), the formulas of Gauss and Weingarten give

$$g(h(JX, Z), JW) = g(Z, W) g(\alpha^{\#}, X) \tag{3.3.2}$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$.

On the other hand, we know that

$$\nabla_Z X = \nabla_X Z = (Z ln f) X \tag{3.3.3}$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$ [7].

Now, let $h^{\top}$ (resp. $A^{\top}$) be the second fundamental form (resp. the shape operator) of $N_{\top}$ in M_1 that is,

$$\nabla_X Y = \nabla^{\top}_X Y + h^{\top}(X,Y), \quad \nabla_X Z = -A^{\top}_Z X + \nabla^{\top}_X Z \tag{3.3.4}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$, where $\nabla^{\top}_X Y$ (resp. $\nabla^{\top}_X Z$) means the tangential (resp. normal) part of $\nabla_X Y$ (resp. $\nabla_X Z$).

Then we have from (3.3.3)

$$g(h^{\top}(X,Y), Z) = g(A^{\top}_Z X, Y) = -(Zlnf)g(X,Y) \tag{3.3.5}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

The equation (3.3.5) means

$$h^{\top}(X,Y) = -(\nabla lnf)g(X,Y) \tag{3.3.6}$$

for any $X, Y \in \mathcal{D}$, where ∇lnf is the gradient of lnf.

Next, we have

Theorem 3.3.1 [11]. In a proper warped product CR-submanifold M_1 of an l.c.K. manifold $\bar{M}$, if the Lee vector field $\alpha^{\#}$ is normal to $\mathcal{D}^{\perp}$, then M_1 is CR-product and trivial.

Proof. Let $\hat{h}$ be the second fundamental form of $N_{\top}$ in $\bar{M}$, that is,

$$\bar{\nabla}_X Y = \nabla^{\top}_X Y + \hat{h}(X,Y)$$

for any $X, Y \in \mathcal{D}$.

Then we have

$$\hat{h}(X,Y) = h^{\top}(X,Y) + h(X,Y) \tag{3.3.7}$$

for any $X, Y \in \mathcal{D}$.

Using (3.3.5) and (3.3.7), we get

$$g(\hat{h}(X,Y), Z) = g(h^{\top}(X,Y), Z) = -(Zlnf)g(X,Y) \tag{3.3.8}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Since, $N_{\top}$ is holomorphic, using (3.2.2) and (3.3.7), we get

$$g(\hat{h}(X, JY), U) = g(J\hat{h}(X,Y), U) + g(JX, Y)g(\alpha^{\#}, U) - g(X,Y)g(\beta^{\#}, U)$$

for any $X, Y \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp} \oplus \Gamma T^{\perp}(M_1)$.

From the above equation, we get

$$g(\hat{h}(X, JY), U) = g(\hat{h}(Y, JX), U) + 2g(JX, Y)g(\alpha^{\#}, U) \tag{3.3.9}$$

for any $X, Y \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp} \oplus \Gamma T^{\perp}M$.

In (3.3.9), we put $Y = JX$, then we have

$$-g(\hat{h}(X, X), U) = g(\hat{h}(JX, JX), U) + 2\|X\|^2 g(\alpha^{\#}, U) \tag{3.3.10}$$

for any $X, Y \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp} \oplus \Gamma T^{\perp}(M)$.

In (3.3.10), if the vector field U is an element of $\mathcal{D}^{\perp}$ (put it Z), then we have

$$--g(\hat{h}(X, X), Z) = g(\hat{h}(JX, JX), Z) + 2\|X\|^2 g(\alpha^{\#}, Z)$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

From (3.3.8) and the above equation, we have

$$Z ln f = g(\alpha^{\#}, Z) \tag{3.3.11}$$

for any $Z \in \mathcal{D}^{\perp}$ if M_1 is proper.

Now for the CR-warped product $M_2 = N_{\top} \times_f N_{\perp}$, we have

Proposition 3.3.2 [11]. In a proper CR-warped product $M_2 = N_{\top} \times_f N_{\perp}$ in an l.c.K. manifold $\bar{M}$, the Lee vector field $\alpha^{\#}$ is orthogonal to $\mathcal{D}^{\perp}$.

Proof. From (3.2.2), we have

$$g(h(X, Y), JZ) = g(\bar{\nabla}_X Y, JZ) = g(X, Y)g(\beta^{\#}, Z) + g(\alpha^{\#}, Z)g(JY, X)$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Since, the left hand side of the above equation is symmetric with respect to X and Y, we get

$$g(JX, Y)g(\alpha^{\#}, Z) = 0.$$

This means

$$g(\alpha^{\#}, Z) = 0$$

for any $Z \in \mathcal{D}^{\perp}$.

Which proves the proposition.

From the proof of the Proposition 3.3.2, we have

$$g(h(X,Y), JZ) = g(A_{JZ}X, Y) = g(X,Y)g(\beta^{\#}, Z) \tag{3.3.12}$$

Especially, if the Lee vector field $\alpha^{\#}$ is tangent to M_2. Then (3.3.12) gives

$$g(h(\mathcal{D},\mathcal{D}), J\mathcal{D}^{\perp}) = \{0\}. \tag{3.3.13}$$

Further, for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$, we have from (3.2.2) and (3.3.3)

$$g(h(JX,Z), JW) = g(\bar{\nabla}_Z JX, JW) = (X\ln f - g(\alpha^{\#}, X))g(Z,W). \tag{3.3.14}$$

Next, we consider the case of $h(\mathcal{D},\mathcal{D}^{\perp}) \subset J\mathcal{D}^{\perp}$. Then we prove that

$$\nabla^{\perp}_X(JZ) = J\nabla_X Z \tag{3.3.15}$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

The proof of (3.3.15) is given as follows:

For any $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$ and $\nu \in \xi$, we have

(i) $g(\nabla^{\perp}_X JZ, JW) = g(J\nabla_X Z, JW) = g(\nabla_X Z, W)$,

(ii) $g(\nabla^{\perp}_X JZ - J\nabla_X Z, \xi) = 0$.

For (i), we have

$$\begin{aligned} g(\nabla^{\perp}_X JZ, JW) &= g(\bar{\nabla}^{\perp}_X JZ, JW) + g(A_{JZ}X, JW) \\ &= g(\bar{\nabla}_X JZ, JW) \\ &= g((\bar{\nabla}_X J)Z, JW) + g(J\bar{\nabla}_X Z, JW). \end{aligned} \tag{3.3.16}$$

By using (3.3.16), we have

$$g((\bar{\nabla}_X J)Z, JW) = 0 \tag{3.3.17}$$

Using (3.3.17) in (3.3.16), we get

$$g(\nabla^{\perp}_X JZ, JW) = g(J\bar{\nabla}_X Z, JW) = g(\bar{\nabla}_X Z, W) = g(\nabla_X Z, W).$$

Similarly, we have (ii).

Thus from (3.3.3) and (3.3.15), we have

Proposition 3.3.3 [11]. In a proper CR-warped product $M_2 = N_\top \times_f N_\perp$ of an l.c.K. manifold $\bar{M}$, if the second fundamental tensor h satisfies

$$h(\mathcal{D}, \mathcal{D}^\perp) \subset J\mathcal{D}^\perp,$$

then

$$\nabla_X^\perp (J\mathcal{D}^\perp) \subset J\mathcal{D}^\perp.$$

Next, we have

Proposition 3.3.4 [11]. In a proper CR-warped product $M_2 = N_\top \times_f N_\perp$ of an l.c.K. manifold $\bar{M}$,

$$g(h(X,Z), JW) = 0$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$ if and only if

$$JXlnf + g(\beta^\#, X) = 0.$$

Proof. In (3.3.14), we replace X to JX, then we have

$$-g(h(X,Z), JW) = (JXlnf + g(\beta^\#, X))g(Z,W).$$

From the above equation, we get the result.

Next, let $h^\perp$ be the second fundamental form of $N_\perp$ in M_2, then we can put

$$\nabla_Z W = \nabla_Z^\perp W + h^\perp(Z,W)$$

for any $Z, W \in \mathcal{D}^\perp$, where $\nabla_Z^\perp W$ is the $\mathcal{D}^\perp$ component of $\nabla_Z W$.

By virtue of (3.3.3) and the above equation, we get

$$h^\perp(Z,W) = -(\nabla lnf)(Z,W). \tag{3.3.18}$$

Thus we have

Proposition 3.3.5. In a proper CR-warped product $M_2 = N_\top \times_f N_\perp$ of an l.c.K. manifold $\bar{M}$, $N_\perp$ is totally umbilical in M.

Corollary 3.3.1. In a proper CR-warped product $M_2 = N_\top \times_f N_\perp$ of an l.c.K. manifold $\bar{M}$, $N_\perp$ is totally geodesic in M_2 that is,

$$h^\perp(\mathcal{D}^\perp, \mathcal{D}^\perp) = \{0\}$$

if and only if the submanifold M_2 is trivial.

3.4. Doubly Warped Product CR-Submanifolds in l.c.K. manifolds

The purpose of this section is to study CR-submanifolds M in a locally conformal Kaehler manifold $\bar{M}$ which are doubly warped products of the form $M ={}_f N_\top \times_b N_\perp$, where $N_\top$ is a holomorphic submanifold and $N_\perp$ is a totally real submanifold in $\bar{M}$. Here $N_\top$ and $N_\perp$ are both totally umbilical in M. If we denote by $h^\top$ and $h^\perp$ the second fundamental forms of $N_\top$ and $N_\perp$, respectively in $\bar{M}$, then

$$\left.\begin{aligned} h^\top(X,Y) &= -\frac{f^2}{b^2} g_{N_\top}(X,Y)\nabla^{N_\perp}(lnf), \\ h^\perp(Z,W) &= -\frac{b^2}{f^2} g_{N_\perp}(Z,W)\nabla^{N_\top}(lnb) \end{aligned}\right\} \tag{3.4.1}$$

for all X, Y tangent to $N_\top$ and Z, W tangent to $N_\perp$.

Lemma 3.4.1 [38]. For a doubly warped product CR-submanifold $M ={}_f N_\top \times_b N_\perp$ in a l.c.K. manifold $\bar{M}$, we have

(1) $g(h(X,Y), JZ) = -\frac{1}{2} g(X,Y) g(JZ, \alpha^\#)$ for all $X, Y \in \mathcal{D},\ Z \in \mathcal{D}^\perp$,

(2) $(\alpha^\#)^{\mathcal{D}^\perp} = \frac{2}{b^2}\nabla^{N_\perp}(lnf)$,

(3) $g(h(\mathcal{D},\mathcal{D}), J\mathcal{D}^\perp) = 0$,

whenever the Lee vector field $\alpha^\#$ is tangent to M.

Proof. Since $\bar{M}$ is l.c.K. , we get

$$g(h(X,Y), JZ) = g(JY, \nabla_X Z) - \frac{1}{2} g(X,Y) g(JZ, \alpha^\#) - \frac{1}{2} g(X, JY) g(Z, \alpha^\#)$$

for any vector field X, Y tangent to $N_\top$ and Z tangent to $N_\perp$.

Combining with (3.2.10), we obtain

$$g(h(X,Y), JZ) = -\frac{1}{2} g(X,Y) g(JZ, \alpha^\#) + g(X, JY)\left(Z(lnf) - \frac{1}{2} g(Z, \alpha^\#)\right)$$

Now, using the symmetry of h and the definition of the gradient i.e.,

$$Z(lnf) = g_{N_\perp}(Z, \nabla^{N_\perp}(lnf)).$$

We get statements (1) and (2). The last statement is a direct consequence of (1).

Next, we have

Lemma 3.4.2 [38]. For a doubly warped product CR-submanifold $M =_f N_\top \times_b N_\perp$ in a l.c.K. manifold $\bar{M}$, we have

$$g(h(JX,Z),JW) = \left(X(lnb) - \frac{1}{2}g(X,\alpha^{\#})\right) g(Z,W) \tag{3.4.2}$$

for any $X \in \mathcal{D}$ and $Z,W \in \mathcal{D}^\perp$.

Proof. Using (3.2.2), we get

$$g(h(JX,Z),JW) = g(\nabla_X Z,W)\frac{1}{2}g(X,\alpha^{\#})g(Z,W)$$

for all X tangent to $N^\top$ and Z,W tangent to $N^\perp$.

Since, M is doubly warped product, from (3.2.10) we get the result.

3.5. A General Inequality for Warped Product CR-Submanifolds in l.c.K. Manifolds

In this section, we calculate the length $\|h\|$ of the second fundamental form h of a warped product CR-submanifold $M_1 = N_\perp \times_f N_\top$ and a CR-warped product $M_2 = N_\top \times_f N_\perp$ of an l.c.K. manifold $\bar{M}$ under the assumption that the Lee vector field $\alpha^{\#}$ is tangent to M_2.

Now, we put dim $\bar{M} = 2m$, dim $M_1 =$ dim $M_2 = n$, dim $N_\top = 2h$ and dim $N_\perp = p$ $(2h + p = n)$.

Let $\{e_1, ..., e_h, e_1^*, ..., e_h^*\}$, $\{e_{2h+1}, ..., e_n\}$, $\{e_{2h+1}^*, ..., e_n^*\}$ and $\{e_{n+p+1}, ..., e_{2m}\}$ be the local orthonormal basis of $\mathcal{D}$, $\mathcal{D}^\perp$, $J\mathcal{D}^\perp$ and ν, respectively, where $e_i^* = Je_i$ for $i \in \{1, 2, ..., h\}$ and $e_{2h+i}^* = Je_{2h+i}$ for $i \in \{1, 2, ..., p\}$.

Remark 3.5.1. It is known that the dimension of the holomorphic submanifold is even.

Now, we have a theorem:

Theorem 3.5.1 [11]. The length $\|h\|$ of the second fundamental form h of a non-trivial proper warped product CR-submanifold M_1 in an l.c.K. manifold $\bar{M}$ satisfies

$$\|h\|^2 \geq 2\left\{h \sum_{l=2h+1}^{n} \left(g(\beta^{\#}, e_l)\right)^2 + p\sum_{i=1}^{2h} \left(g(\beta^{\#}, e_i)\right)^2\right\}. \tag{3.5.1}$$

Proof. The length $\|h\|$ of the second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (g(h(e_i, e_j), e_r))^2 . \tag{3.5.2}$$

The equation (3.5.2) is written as

$$\begin{aligned}
\|h\|^2 &= \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{n} (g(h(e_i, e_j), e_r))^2 + \sum_{r=n+p+1}^{2m} \sum_{i,j=1}^{n} (g(h(e_i, e_j), e_r))^2 . \\
&\geq \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{n} (g(h(e_i, e_j), e_r))^2 = \sum_{l=2n+1}^{n} \sum_{i,j=1}^{n} (g(h(e_i, e_j), Je_l))^2 . \\
&= \sum_{l=2h+1}^{n} \left\{ \sum_{i,j=1}^{2h} (g(h(e_i, e_j), e_l^*))^2 + 2 \sum_{i=1}^{2h} \sum_{j=2h+1}^{n} (g(h(e_i, e_j), e_l^*))^2 \right. \\
&\quad \left. + \sum_{i,j=2h+1}^{n} (g(h(e_i, e_j), e_l^*))^2 \right\} .
\end{aligned}$$

Let M_1 be a non-trivial proper warped product CR-submanifold in an l.c.K. manifold $\bar{M}$. Then we know that

$$Z ln f = g(\alpha^{\#}, Z) \tag{3.5.3}$$

for any $Z \in \mathcal{D}^{\perp}$.

In virtue of (3.2.7) and (3.5.3), we have

$$g(h(X, Y), JZ) = g(\beta^{\#}, Z) g(X, Y) \tag{3.5.4}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

The equation (3.5.4) means

$$g(h(e_i, e_j), e_l{}^*) = g(\beta^{\#}, e_l)\delta_{ij}$$

for $i, j \in \{1, ..., 2h\}$ and $l \in \{2h + 1, ..., n\}$. So, we get

$$\sum_{l=2h+1}^{n} \sum_{i,j=1}^{2h} (g(h(e_i, e_j), e_l{}^*))^2 = 2h \sum_{l=2h+1}^{n} (g(\beta^{\#}, e_l))^2. \tag{3.5.5}$$

Next, we have from (3.3.2),

$$g(h(X, Z), JW) = g(\beta^{\#}, X) g(Z, W) \tag{3.5.6}$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^{\perp}$.

Using (3.5.6), we get

$$g(h(e_i, e_j), Je_k) = g(\beta^{\#}, e_i)\delta_{jk}$$

for $i \in \{1, ..., 2h\}$ and $j, k \in \{2h+1, ..., n\}$. Thus we have

$$\sum_{i=1}^{2h} \sum_{j,k=2h+1}^{n} (g(h(e_i, e_j), e_k{}^*))^2 = \sum_{i=1}^{2h} \sum_{j,k=2h+1}^{n} \{g(\beta^{\#}, e_i)\delta_{jk}\}^2 = p\sum_{i=1}^{2h}(g(\beta^{\#}, e_i))^2. \tag{3.5.7}$$

By virtue of (3.5.5) and (3.5.7)

$$\|h\|^2 \geq 2h \sum_{l=2h+1}^{n} (g(\beta^{\#}, e_l))^2 + 2p\sum_{i=1}^{2h}(g(\beta^{\#}, e_i))^2 + \sum_{i,j,l=2h+1}^{n} (g(h(e_i, e_j), e_l{}^*))^2. \tag{3.5.8}$$

Hence proved.

Remark 3.5.2. The equality in (3.5.1) is satisfied if and only if

$$g(h(\Gamma T(M), \Gamma T(M)), \nu) = \{0\} \ , \ g(h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}), J\mathcal{D}^{\perp}) = \{0\}. \tag{3.5.9}$$

Next, we have a theorem

Theorem 3.5.2 [11]. Let $M_2 = N_T \times_f N_{\perp}$ be a proper CR-warped product in an l.c.K. manifold $\bar{M}$. If the Lee vector field $\alpha^{\#}$ is tangent to M_2, the length $\|h\|$ of the second fundamental form h satisfies

$$\|h\|^2 \geq \ 2p\sum_{i=1}^{2h} \left(Je_i lnf - g(\alpha^{\#}, Je_i)\right)^2 \tag{3.5.10}$$

and the equality is satisfied if and only if the second fundamental form h satisfies

$$h(\mathcal{D}, \mathcal{D}) = \{0\}, \ h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) = \{0\}, \ h(\mathcal{D}, \mathcal{D}^{\perp}) \subset J\mathcal{D}^{\perp}. \tag{3.5.11}$$

Proof. Let M_2 be a proper CR-warped product in an l.c.K. manifold $\bar{M}$ and the Lee vector field $\alpha^{\#}$ be tangent to M_2. Then by Proposition 3.3.2, the Lee vector field $\alpha^{\#}$ is in $\mathcal{D}$.

On the other hand, we have from Proposition 3.3.2 and (3.3.12)

$$g(h(e_i, e_j), e_l{}^*) = g(e_i, e_j)g(\beta^{\#}, e_l) = -\delta_{ij}\ g(\alpha^{\#}, e_l{}^*) = 0$$

for $i, j \in \{1, ..., 2h\}$ and $l \in \{2h+1, ..., n\}$.

This means that
$$g(h(\mathcal{D},\mathcal{D}),J\mathcal{D}^{\perp})=\{0\}. \tag{3.5.12}$$
So, we have
$$\|h\|^2\geq\sum_{l=2h+1}^{n}\left\{2\sum_{i=1}^{2h}\sum_{j=2h+1}^{n}(g(h(e_i,e_j),e_l{}^*))^2+\sum_{i,j=2h+1}^{n}(g(h(e_i,e_j),e_l{}^*))^2\right\}$$
$$=2\sum_{i=1}^{2h}\sum_{j,l=2h+1}^{n}(g(h(e_i,e_j),e_l{}^*))^2+\sum_{i,j,l=2h+1}^{n}(g(h(e_i,e_j),e_l{}^*))^2.$$
Next by virtue of (3.3.14), we get
$$g(h(e_i,e_j),e_l{}^*)=-(Je_i lnf-g(\alpha^\#,Je_i))\delta_{jl}$$
for $i\in\{1,....,2h\}$ and $j,l\in\{2h+1,.....,n\}$. So, we have
$$\sum_{j,l=2h+1}^{n}(g(h(e_i,e_j),e_l^*))^2=(Je_i lnf-g(\beta^\#,e_i))^2p. \tag{3.5.13}$$
Thus from (3.5.13), we have
$$\|h\|^2\geq 2\sum_{i=1}^{2h}\{(Je_i lnf-g(\alpha^\#,Je_i))^2p\}+\sum_{i,j,l=2h+1}^{n}(g(h(e_i,e_j),e_l{}^*))^2$$
$$\geq 2p\sum_{i=1}^{2h}(Je_i lnf-g(\alpha^\#,Je_i))^2. \tag{3.5.14}$$
Thus we have
$$\|h\|^2\geq 2p\sum_{i=1}^{2h}\left(Je_i lnf-g(\alpha^\#,Je_i)\right)^2. \tag{3.5.15}$$
Next, let us consider the equality case of (3.5.15). Then we have
$$g(h(\Gamma T(M),\Gamma T(M)),\nu)=\{0\},\quad g(h(\mathcal{D}^{\perp},\mathcal{D}^{\perp}),J\mathcal{D}^{\perp})=\{0\}.$$
By virtue of (3.5.12) and the above relations, we get
$$h(\mathcal{D},\mathcal{D})=\{0\},\qquad h(\mathcal{D}^{\perp},\mathcal{D}^{\perp})=\{0\},\qquad h(\mathcal{D},\mathcal{D}^{\perp})\subset J\mathcal{D}^{\perp}. \tag{3.5.16}$$
Hence proved.

From (3.3.18) and (3.5.16), We have

Theorem 3.5.3 [11]. Let $M_2=N_\top\times_f N_\perp$ be a proper CR-warped product in an l.c.K. manifold $\bar{M}$. If we assume inequality in (3.5.10), we get

(i) N_T is totally geodesic in $\bar{M}$,

(ii) $N_\perp$ is totally umbilical in $\bar{M}$.

Next, we have

Theorem 3.5.4 [11]. Let $M_2 = N_T \times_f N_\perp$ be an anti-holomorphic CR-warped product in an l.c.K. manifold $\bar{M}$ and Lee vector field $\alpha^\#$ is tangent to M_2. If dim $N_\perp > 1$, then equality in (3.5.10) holds if and only if the totally real submanifold $N_\perp$ is totally umbilical in $\bar{M}$. If dim $N_\perp = 1$, then M_2 is a real hypersurface of $\bar{M}$ and the equality in (3.5.10) holds if the vector field $J\xi$ ($\xi \in \mathcal{D}^\perp$) is a principal vector field with zero principal curvature and then the Lee form α and the warping function f satisfies the following relation

$$\alpha_i = \frac{\partial lnf}{\partial x_i} \tag{3.5.17}$$

where $\{x^1, x^2, ..., x^{2h}\}$ is a local co-ordinate system on $\mathcal{D}$ and $\alpha = \alpha_i dx^i$.

Now from [34], we have

Proposition 3.5.1 [34]. A holomorphic submanifold M in an l.c.K. manifold $\bar{M}$ such that the Lee vector field $\alpha^\#$ is tangent to M is an l.c.K. manifold.

From (3.2.3) and (3.5.11) and the above proposition, we get

Theorem 3.5.5 [11]. Let $M_2 = N_T \times_f N_\perp$ be a proper CR-warped product such that equality holds in (3.5.10) and Lee vector field $\alpha^\#$ is tangent to M_2 in an l.c.K. space form $\bar{M}(c)$. Then $N_\perp$ is an l.c.K. space form with constant holomorphic sectional curvature c.

Theorem 3.5.6 [11]. Let $M_2 = N_T \times_f N_\perp$ be a proper CR-warped product such that equality holds in (3.5.10), in an l.c.K. space form $\bar{M}(c)$. Then if dim $N_\perp > 2$, then $N_\perp$ is a real space form with constant scalar curvature

$$\frac{1}{4}\left\{c + 4\|H\|^2 + 3\left(\frac{1}{2}\|\alpha\|^2 + \alpha(A)\right)\right\}.$$

CHAPTER 4

WARPED AND DOUBLY WARPED PRODUCT CONTACT CR-SUBMANIFOLDS IN TRANS-SASAKIAN MANIFOLDS

4.1. Introduction

In this chapter we study the existence of contact CR-warped product submanifolds in more general setting of trans-Saskian manifold and the non existence of the proper doubly warped product contact CR-submanifolds. Also we study the semi-slant submanifolds in trans-Sasakian manifold.

The study of warped product submanifolds was initiated by R.L.Bishop and B.O.Niell [6] with differential geometric point of view and the study of slant immersions was initiated by B.Y.Chen [19]. A.Lotta [33] extended the notion to the setting of almost contact metric manifolds. In fact, semi slant submanifolds in almost Hermitian manifolds are defined on the lines of CR-submanifolds. In the setting of almost contact metric manifold, semi slant submanifolds are defined and investigated by J.L.Cabrerizo et.al.[20]. These submanifolds are studied in further specialized setting of K-contact and Sasakian manifolds by J.L.Cabrerizo et.al.[20]. They obtained the integrability conditions of the distributions on these submanifolds and studied the geometry of the leaves of these distributions. In [31], V.A.Khan and Meraj Ali Khan have extended the study of the semi slant submanifolds to the setting of trans-Sasakian manifolds.

After the impulse given by B.Y.Chen [15], [16], the study of warped product CR-submanifolds in Kaehler manifolds was extensively done only since 2001. After it, another line of research similar to that concerning Sasakian geometry as the odd dimensional version of Kaehlerian geometry was developed, namely warped product contact CR-submanifolds in Saskian manifolds (cf.,[39], [40]). In [41] the another has proved the non-existence of the proper doubly warped product contact CR-submanifolds in trans-Sasakian manifolds.

4.2. Semi-slant Submanifolds in Trans-Sasakian Manifolds

The purpose of this section is to study the semi-slant submanifolds in the setting of trans-Sasakian manifolds. N.Papaghuic [45] introduced the notion of semi-slant submanifold of an almost Hermitian manifold. A semi-slant submanifold is a generalized version of CR-submanifold. Recently, J.L.Cabrerizo et.al.[20] gave the contact version of semi-slant submanifold and have found several interesting results in this setting.

First, we discuss some preliminaries of trans-Sasakian manifold.

Let $\bar{M}$ be an odd dimensional Riemannian manifold with a Riemannian metric g and Riemannian connection $\bar{\nabla}$. Then $\bar{M}$ is said to be an almost contact metric manifold [10] if there exist on $\bar{M}$, a tensor ϕ of the type (1,1), a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

$$(a)\ \phi^2X = -X + \eta(X)\xi, \qquad (b)\ g(X,\xi) = \eta(X), \tag{4.2.1}$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta c\phi = 0, \tag{4.2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{4.2.3}$$

for any $X, Y \in T(\bar{M})$. In this case

$$g(\phi X, Y) = -g(X, \phi Y) \tag{4.2.4}$$

Throughout, all the maps are assumed to be differentiable. The fundamental 2-form Φ on $\bar{M}$ is given by

$$\Phi(X,Y) = g(X, \phi Y).$$

Moreover,

$$(\bar{\nabla}_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{4.2.5}$$

then $\bar{M}$ is said to be trans-Sasakian manifold [43], where X and Y are tangential vector fields on $\bar{M}$ and α, β are smooth functions on $\bar{M}$. If α (resp. β) is zero then $\bar{M}$ is called β-Kenmotsu (resp. α-Sasakian). If α and β both are zero then the manifold $\bar{M}$ is called Cosymplectic.

On a trans-Sasakian manifold $\bar{M}$, we have

$$\bar{\nabla}_X\xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi) \tag{4.2.6}$$

From (4.2.4) and (1.4.21), it can be easily obtain that for each $x \in M$ and $X, Y \in T_x(M)$

$$g(TX, Y) = -g(X, TY) \tag{4.2.7}$$

and therefore $g(T^2X, Y) = g(X, T^2Y)$ which implies that the endomorphism $T^2 = Q$ is self adjoint. Moreover, it can be seen that the eigen value of Q belong to [-1,0] and that each non vanishing eigen values of Q has even multiplicity.

We have the covariant derivative of T, Q and N as follows:

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{4.2.8}$$

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y, \tag{4.2.9}$$

$$(\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y \tag{4.2.10}$$

for any $X, Y \in T(M)$.

Throughout, the vector field ξ is assumed to be tangential to M, for otherwise M is simply anti-invariant (cf., [33]). Using (4.2.6) and (1.3.2), we have the following

$$(a)\ \nabla_X\xi = -\alpha TX + \beta(X - \eta(X)\xi), \qquad (b)\ h(X,\xi) = -\alpha NX \tag{4.2.11}$$

and by using (4.2.5),(1.3.2),(1.3.3),(1.3.4),(1.4.22),(4.2.8) and (4.2.10), we obtain

$$(\nabla_X T)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)TX) + A_{NY}X + th(X,Y), \tag{4.2.12}$$

$$(\nabla_X N)Y = -\beta\eta(Y)NX - h(X,TY) + nh(X,Y). \tag{4.2.13}$$

Now, we discuss some results on the integrability conditions of the distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ on a semi-slant submanifold of trans-Sasakian manifold.

We have

Lemma 4.2.1. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then

$$g([X,Y],\xi) = 2\alpha g(TX,Y) \tag{4.2.14}$$

for any $X, Y \in \mathcal{D} \oplus \mathcal{D}^\theta$.

As a consequence of Lemma 4.2.1, we have

Corollary 4.2.1. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the distributions $\mathcal{D}$ and $\mathcal{D} \oplus \mathcal{D}^\theta$ are not integrable on M, in general.

In particular, if $\bar{M}$ is a β-Kenmotsu manifold, we have

Corollary 4.2.2 [31]. The distribution $\mathcal{D} \oplus \mathcal{D}^\theta$ on a semi-slant submanifold of a β-Kenmotsu manifold is integrable. Where as $\mathcal{D}$ is integrable if and only if

$$h(X,\phi Y) = h(\phi X, Y)$$

for each $X, Y \in \mathcal{D}$.

Proof. For any $X,\ Y \in \mathcal{D}$ and $V \in T^{\perp}(M)$,

$$g(\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X, V) = g(h(X,\phi Y) - h(\phi X, Y), V)$$

which on using equation (4.2.5) and (1.4.32) yields

$$g(NP_2[X,Y],V) = g(h(X,\phi Y) - h(\phi X,Y),V). \tag{4.2.15}$$

The corollary follows in view of (4.2.14) and (4.2.15).

Next, for the semi-slant distribution, we have

Proposition 4.2.1 [31]. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the slant distribution $\mathcal{D}^\theta$ is integrable if and only if slant angle of $\mathcal{D}^\theta$ is $\pi/2$ i.e., $\mathcal{D}^\theta$ is anti-invariant.

Proof. For any $Z, W \in \mathcal{D}^\theta$ by (4.2.14), we have

$$g([Z,W],\xi) = 2\alpha g(W,TZ).$$

If $\mathcal{D}^\theta$ is integrable, then $T\mid_{\mathcal{D}^\theta} = 0$ that means $\theta = \pi/2$.

Conversely, if $sla(\mathcal{D}^\theta) = \pi/2$ then $\phi Z = NZ$ for each Z in $\mathcal{D}^\theta$ and by equations (4.2.5), (1.3.2) and (1.3.3),

$$\phi\nabla_Z W + \phi h(Z,W) = -A_{NW}Z + \nabla_Z^\perp NW - \alpha g(Z,W)\xi$$

for each $Z, W \in \mathcal{D}^\theta$.

Interchanging W and Z in the above equation and subtracting the obtained relation from the same , we get

$$\phi[Z,W] = A_{NZ}W - A_{NW}Z + \nabla_Z^\perp NW - \nabla_W^\perp NZ. \tag{4.2.16}$$

Further, by using equations (1.3.2), (1.3.3), (1.3.4), (4.2.4) and (4.2.5), we get

$$A_{NZ}W = A_{NW}Z \tag{4.2.17}$$

for each Z, W in $\mathcal{D}^\theta$.

Now equation (4.2.16) in view of equations (4.2.14), (4.2.1) and (4.2.17) gives

$$[Z,W] = \phi(\nabla_Z^\perp NW - \nabla_W^\perp NZ). \tag{4.2.18}$$

The right hand side of the above lies in $\mathcal{D}^\theta$ because on using equations (4.2.5), (4.2.7) and (1.3.2), we see that

$$g(V,\nabla_W^\perp NZ) = -g(A_{\phi V}W,Z)$$

for all $V \in \mu$ and $Z, W \in \mathcal{D}^\theta$. This shows that

$$g(\nabla_Z^\perp NW - \nabla_W^\perp NZ, V) = 0$$

i.e., $\nabla^{\perp}_{Z}NW - \nabla^{\perp}_{W}NZ$ lies in $N\mathcal{D}^{\theta}$ for each Z, W in $\mathcal{D}^{\theta}$.

Thus from equation (4.2.18),

$$[Z, W] \in \mathcal{D}^{\theta}$$

Corollary 4.2.3 [31]. Let M be a semi-slant submanifold of a β-Kenmotsu manifold $\bar{M}$. Then the slant distribution $\mathcal{D}^{\theta}$ is integrable if and only if

$$P_1(\nabla_Z TW - A_{NW}Z - \nabla_W TZ + A_{NZ}W) = 0$$

for all $Z, W \in \mathcal{D}^{\theta}$.

Proof. By using equations (1.3.2), (1.3.3) and (4.2.5), we have

$$g(T[Z, W], X) = g(\nabla_Z TW - A_{NW}Z - \nabla_W TZ + A_{NZ}W, X) \tag{4.2.19}$$

for any $Z, W \in \mathcal{D}^{\theta}$ and $X \in \mathcal{D}$.

The corollary follows by the equations (4.2.14) and (4.2.19).

By equation (4.2.5), for any $Y \in T(M)$

$$(\bar{\nabla}_{\xi}\phi)Y = 0$$

i.e.,

$$\bar{\nabla}_{\xi}\phi Y = \phi\bar{\nabla}_{\xi}Y. \tag{4.2.20}$$

In particular, for $X \in \mathcal{D}$, equation (4.2.20) together with equations (1.3.2) and (4.2.11) implies that

$$\nabla_{\xi}X \in \mathcal{D}. \tag{4.2.21}$$

The above observation together with (4.2.11)(a), yields

Lemma 4.2.2. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$ then

$$[X, \xi] \in \mathcal{D} \quad \text{and} \quad [Z, \xi] \in \mathcal{D}^{\theta}$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\theta}$.

This leads to the following:

Proposition 4.2.2 [31]. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then

(*i*) $\mathcal{D} \oplus < \xi >$ is integrable if and only if

$$h(X, \phi Y) = h(Y, \phi X),$$

(*ii*) $\mathcal{D}^\theta \oplus <\xi>$ is integrable if and only if

$$P_1(\nabla_Z TW - A_{NW}Z - \nabla_W TZ + A_{NZ}W) = 0$$

for each X, Y in $\mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$.

The Nijenhuis tensor field N with respect to T is given by

$$N(X,Y) = [TX, TY] + T^2[X,Y] - T[X,TY] - T[TX,Y]$$

for all $X, Y \in T(M)$.

In particular, for $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\theta$,

$$N(X,Z) = (\bar{\nabla}_{TX}T)Z - (\bar{\nabla}_{TZ}T)X + T(\bar{\nabla}_Z T)X - T(\bar{\nabla}_X T)Z.$$

Using equation (4.2.12) the above equation becomes

$$N(X,Z) = A_{NZ}TX + th(TX,Z) - th(TZ,X) - TA_{NZ}X. \qquad (4.2.22)$$

Lemma 4.2.3 [31]. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. If the distribution $\mathcal{D} \oplus <\xi>$ is integrable and its leaves are totally geodesic in M, then

(*i*) $h(\mathcal{D}, \mathcal{D}^\theta) \in \mu$,

(*ii*) $N(\mathcal{D}, \mathcal{D}^\theta) \in \mathcal{D}^\theta$.

Proof. By the hypothesis,

$$g(\nabla_X Y, Z) = 0$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\theta$ which gives

$$g(\phi\bar{\nabla}_X Y, \phi Z) = 0.$$

On using (4.2.5), (1.3.2) and (1.3.4), we get

$$g(h(X, \phi Y), NZ) = 0.$$

This proves the statement (*i*).

To prove statement (*ii*), use (4.2.22) to get

$$g(N(X,Z),Y) = g(A_{NZ}TX + th(TX,Z) - th(TZ,X) - TA_{NZ}X, Y).$$

The right hand side of the above equation is zero in view of statement (*i*) and the lemma is proved completely.

Corollary 4.2.4. If the invariant distribution $\mathcal{D}$ on a semi-slant submanifold M of a β-Kenmotsu is integrable and its leaves are totally geodesic in M, then statement (i) and (ii) of Lemma 4.2.3 are true.

Remark 4.2.1. The above corollary is not true for the setting of semi-slant submanifold of a α-Sasakian manifold, as in this case $\mathcal{D}$ is not integrable (cf.,[4]) where as $\mathcal{D}$ is integrable on a semi-slant submanifold of a Kenmotsu manifold if and only if

$$h(X, \phi Y) = h(\phi X, Y)$$

for each X,Y in $\mathcal{D}$.

Lemma 4.2.4 [31]. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. If the slant distribution $\mathcal{D}^\theta \oplus < \xi >$ is integrable and its leaves are totally geodesic in M, then

(i) $h(\mathcal{D}, \mathcal{D}^\theta) \in \mu$,

(ii) $N(\mathcal{D}, \mathcal{D}^\theta) \in \mathcal{D}$.

Proof. Suppose $\mathcal{D}^\theta \oplus < \xi >$ is integrable and its leaves are totally geodesic, then for any $Z, W \in \mathcal{D}^\theta$ and $X \in \mathcal{D}$

$$g(\nabla_Z X, \varphi X) = 0.$$

Using (4.2.4), (4.2.5), (1.3.2) and (1.3.3) together with above equation, we get

$$g(h(X, Z), NW) = 0.$$

This proves (i).

To prove (ii), on applying formula (4.2.22) and using the statement (i), we get

$$g(h(X, Z), W) = 0$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$.

This proves (ii).

4.3. Warped Product CR-Submanifolds in Trans-Sasakian Manifolds

Throughout this section, it is assumed that $\bar{M}$ is a trans-Sasakian manifold and $M = N_\perp \times_f N_\top$ be a Warped product CR-submanifold of a trans-Sasakian manifold $\bar{M}$ [27]. Such submanifolds are always tangent to the structure vector field ξ. There are two cases;

(i) ξ tangent to $N_\top$,

(ii) ξ tangent to $N_\perp$.

First, we consider ξ tangent to $N_\top$.

Theorem 4.3.1 [27]. Let $\bar{M}$ be a $(2m+1)$-dimensional trans-Sasakian manifold . Then there do not exist warped product CR-submanifolds $M = N_\perp \times_f N_\top$ such that $N_\top$ is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$.

Proof. Assume $M = N_\perp \times_f N_\top$ is a warped product CR-submanifold of a trans-Sasakian manifold such that $N_\top$ is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. By Lemma 2.2.2, we have

$$\nabla_X Z = \nabla_Z X = (Zlnf)X \tag{4.3.1}$$

for any $X \in T(N_\top)$ and $Z \in T(N_\perp)$.

In particular, for $X = \xi$, we have

$$\nabla_Z \xi = (Zlnf)\xi. \tag{4.3.2}$$

Using the structure equation of trans-Sasakian manifold and equations (1.3.2) and (4.3.1), it follows that

$$\begin{aligned} -\alpha\phi Z + \beta(Z - \eta(Z)\xi) &= \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z,\xi) \\ -\alpha FZ + \beta Z &= \nabla_Z \xi + h(Z,\xi), \\ \Rightarrow \quad \nabla_Z \xi = \beta Z, &\quad h(Z,\xi) = -\alpha FZ \end{aligned} \tag{4.3.3}$$

From equations (4.3.2) and (4.3.3), we conclude that

$$Zlnf = 0$$

for all $Z \in T(N_\perp)$ i.e., f is constant for all $Z \in T(N_\perp)$.

This completes the proof of the theorem.

Now, we will discuss the case (ii) i.e., ξ tangent to $N_\perp$.

Theorem 4.3.2 [27]. Let $\bar{M}$ be a $(2m+1)$-dimensional trans-Sasakian manifold. Then there does not exist proper warped product CR-submanifold $M = N_\perp \times_f N_\top$ such that $N_\top$ is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold tangent to ξ of $\bar{M}$, unless $\bar{M}$ is β-Kenmotsu.

Proof. Assume $M = N_\perp \times_f N_\top$ is a warped product submanifold of a trans-Sasakian manifold such that $N_\top$ is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold tangent to ξ of $\bar{M}$.

In view of Lemma (2.2.2), we have

$$\nabla_X Z = \nabla_Z X = (Zlnf)X \tag{4.3.4}$$

for any $X \in T(N_\perp)$ and $Z \in T(N_\perp)$.

In particular, $Z = \xi$, we have

$$\nabla_X \xi = (\xi lnf)X. \tag{4.3.5}$$

From (4.2.1), (4.2.2), (4.2.3), (4.2.5), (4.2.6), (1.3.2) and (4.3.5), we get

$$(\xi lnf)X = -\alpha\phi X + \beta X \quad , \quad h(X,\xi) = 0. \tag{4.3.6}$$

Since, X and ϕX are linearly independent, from the above equation we see that proper warped product CR-submanifold of $\bar{M}$ are possible only if $\alpha = 0$ and $\beta = \xi lnf$ and in this case the ambient manifold becomes a Kenmotsu manifold, known as β-Kenmotsu.

This completes the proof.

4.4. CR-Warped Product Submanifolds in Trans-Sasakian Manifolds

In this section, we study CR-warped product submanifolds $N_\top \times_f N_\perp$. There are also two cases which are

(*i*) ξ tangent to $N_\perp$,

(*ii*) ξ tangent to $N_\top$.

We start with the case (*i*)

Theorem 4.4.1 [27]. Let $\bar{M}$ be a $(2m+1)$-dimensional trans-Sasakian manifold. Then there does not exist CR-warped product submanifold $M = N_\top \times_f N_\perp$ such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and $N_\top$ is an invariant submanifold of $\bar{M}$.

Proof. Let $M = N_\top \times_f N_\perp$ be a CR-warped product submanifold of a trans-Sasakian manifold such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and $N_\top$ is an invariant submanifold of $\bar{M}$.

Let $X \in T(N_{\mathrm{T}})$ and $\xi \in T(N_{\mathrm{T}})$, then we have

$$\nabla_X \xi = (X \ln f)\xi. \tag{4.4.1}$$

Also, by equation (4.2.6) and (1.3.2), we have

$$\alpha\phi X + \beta(X - \eta(X)\xi) = \bar{\nabla}_X \xi - \nabla_X \xi + h(X,\xi),$$
$$\nabla_X \xi = -\alpha\phi X + \beta X, \qquad h(X,\xi) = 0. \tag{4.4.2}$$

By equation (4.4.1) and (4.4.2), we get

$$(X \ln f)\xi = -\alpha\phi X + \beta X$$

$\Rightarrow$
$$X \ln f = 0$$

for all $X \in T(N_{\mathrm{T}})$.

This means that f is constant for all $X \in T(N_{\mathrm{T}})$.

This completes the proof of the theorem.

Now, we discuss the case (ii) i.e., ξ tangent to N_{T}.

Assume that $M = N_{\mathrm{T}} \times_f N_{\perp}$ be a CR-warped product submanifold of a trans-Sasakian manifold such that N_{T} is an invariant submanifold tangent to ξ and $N_{\perp}$ is an anti-invariant submanifold of $\bar{M}$.

Let $Z \in T(N_{\perp})$ and $\xi \in T(N_{\mathrm{T}})$, then

$$\nabla_Z \xi = (\xi \ln f)Z. \tag{4.4.3}$$

Also by structure equation and (1.3.2), we have

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(X,\xi),$$
$$\nabla_Z \xi = \beta Z, \;\; h(X,\xi) = -\alpha F Z. \tag{4.4.4}$$

From the equation (4.4.3), (4.4.4), we get

$$(\xi \ln f) = \beta \text{ for all } Z \in T(N_{\perp})$$

In this case warped products do exist with smooth function $\beta \in C^{\infty}(N_{\mathrm{T}})$, which is a case similar to Theorem 4.3.2.

4.5. Totally Umbilical Semi-slant Submanifolds in Trans-Sasakian Manifolds

In this section, we study the totally umbilical semi-slant submanifolds in trans-Sasakian manifolds. We start this section with the definition of totally umbilical submanifold

Definition 4.5.1. A submanifold M is said to be totally umbilical submanifold if its second fundamental form satisfies

$$h(X,Y) = g(X,Y)H \tag{4.5.1}$$

for all $X, Y \in T(M)$, where H is the mean curvature vector.

First, we discuss the following preliminary result

Proposition 4.5.1 [31]. Let M be a semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$ with $h(X,TX) = 0$ for each $X \in \mathcal{D}\oplus < \xi >$. If $X \in \mathcal{D}\oplus < \xi >$ is integrable, then each of its leaves is totally geodesic in M as well as in $\bar{M}$.

As an immediate consequence of the above, we have

Corollary 4.5.1. Let M be a totally umbilical semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$. If the $\mathcal{D}\oplus < \xi >$ is integrable, then leaves of $\mathcal{D}\oplus < \xi >$ are totally geodesic in Mas well as in $\bar{M}$.

Theorem 4.5.1 [31]. Let M be a totally umbilical semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$ with dim $\mathcal{D} \neq 0$, then the mean curvature vector field H of M is a global section $N\mathcal{D}^\theta$.

Proof. Let $X \in \mathcal{D}^\theta$ be a unit vector field and $V \in \mu$, then

$$g(H,V) = g(h(X,X),V) = g(\bar{\nabla}_X \phi X, \phi V) = g(h(X,\phi X), \phi V) = 0,$$

which proves the assertion.

Theorem 4.5.2 [31]. Let M be a totally umbilical semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$ with $\alpha \neq 0$ on M. Then M is an invariant submanifold.

Proof. By (4.5.1), $h(Z,\xi) = 0$ for any $Z \in \mathcal{D}^\theta$ and therefore by (4.2.11)(b), $NZ = 0$.

Hence, M is an invariant submanifold.

Consequently, we have

Corollary 4.5.2. There does not exist totally umbilical proper semi-slant submanifold of a trans-Sasakian manifold provided $\alpha \neq 0$ on M.

From Theorem 4.5.1 and 4.5.2, it follows that

Corollary 4.5.3. A totally umbilical semi-slant submanifold of a trans-Sasakian manifold $\bar{M}$ is in fact, totally geodesic in $\bar{M}$.

4.6. Doubly Warped Product CR-Submanifolds in Trans-Sasakian Manifolds

In this section, we discuss the non existence of proper doubly warped product contact CR-submanifolds in trans-Sasakian manifolds.

We have

Theorem 4.6.1 [41]. There is no proper doubly warped product contact CR-submanifold in trans-Sasakian manifold.

Proof. Let $M = _f N_T \times_b N_\perp$ be a doubly warped product contact CR-submanifold in a trans-Sasakian manifold $(\bar{M}, \phi, \xi, \eta, g)$, i.e., N_T is a ϕ-invariant submanifold and $N_\perp$ is a ϕ-anti-invariant submanifold.

Since, either $\xi \in \mathcal{D}$ or $\xi \in \mathcal{D}^\perp$.

Case 1. $\xi \in \mathcal{D}$ i.e., ξ is tangent to N_T.

Taking $Z \in \chi(N_\perp)$, we have

$$\nabla_Z \xi = \beta Z.$$

On the other hand, from (1.4.12), we get

$$\nabla_\xi Z = Z(lnf)\ \xi + \xi(lnb) Z.$$

It follows, since the two distributions are orthogonal, that is

$$\xi(lnb) = \beta\ , \quad Z(lnf) = 0 \tag{4.6.1}$$

for all $Z \in \mathcal{D}^\perp$.

The second condition yields that $f \equiv$ constant, so we can not have doubly warped contact CR-submanifolds of the form $M = _f N_T \times_b N^\perp$, with ξ tangent to N_T, other than warped product contact CR-submanifolds. Moreover, in this case β is a smooth function on N_T.

Case 2. $\xi \in \mathcal{D}^\perp$ i.e., ξ is tangent to $N_\perp$.

Taking $X \in \chi(N_T)$, we have $\nabla_X\xi = -\alpha PX + \beta X$ in one hand and $\nabla_X\xi = X(lnb)\xi + \xi(lnf)X$ in the other hand. Since, the two distribution are orthogonal, we immediately get

$$\xi(lnf)X = -\alpha PX + \beta X \ , \ X(lnb) = 0 \tag{4.6.2}$$

for all $X \in \mathcal{D}$.

The second condition in (4.6.2) shows that M is a CR-warped product between a ϕ-anti invariant manifold $N_\perp$ tangent to the structure vector field ξ and an invariant manifold N_T.

If dim $\mathcal{D} = 0$, then M is a ϕ-anti invariant submanifold in $\bar{M}$. Otherwise, one can choose $X \neq 0$ and thus X and PX are linearly independent.

Using first relation in (4.6.2), one gets $\alpha = 0$ and $\beta = \xi(lnf)$. This means that the ambient manifold is β-Kenmotsu with $\beta \in C^\infty(N_\perp)$.

This ends the proof.

In 1992, J.C.Marero in [42] showed that a trans-Sasakian manifold of dimension $\geq$ 5 is either α-Sasakian, β-Kenmotsu or Cosymplectic. So, we can state

Corollary 4.6.1. Let $\bar{M}$ be

(1) either an α-Sasakian manifold,

(2) or a β-Kenmotsu manifold,

(3) or a Cosymplectic manifold.

Then there is no proper doubly warped product contact CR-submanifolds in $\bar{M}$.

Now, we will discuss an example of warped-product contact CR-submanifold of the type $N_\perp \times_f N_T$ in Kenmotsu manifold, with ξ tangent to $N_\perp$

Example 4.6.1. Consider the complex space C^m with the usual Kaehler structure and real global coordinates $(x^1, x^2, ..., x^m, y^m)$.

Let $\bar{M} = R \times_f C^m$ be the real line R and C^m, where the warping function is $f = e^z$, z being the global coordinate on R. Then $\bar{M}$ is a Kenmotsu manifold.

Consider the distribution

$$\mathcal{D} = span\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, ...\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^s}\right\},$$

and

$$\mathcal{D}^{\perp} = span\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x^{s+1}}, \ldots\ldots \frac{\partial}{\partial x^{m}}\right\}$$

which are integrable and denoted by N_{T} and $N_{\perp}$ the integral submanifolds, respectively. Let

$$g_{N_{\mathrm{T}}} = \sum_{i=1}^{s}((dx^{i})^{2} + (dy^{i})^{2})$$

and

$$g_{N_{\perp}} = dz^{2} + e^{2z}\sum_{a=s+1}^{m}(dx^{a})^{2}$$

be Riemannian metrics on N_{T} and $N_{\perp}$, respectively.

Then $M = N_{\perp} \times_{f} N_{\mathrm{T}}$ is a contact CR-submanifold isometrically immersed in $\bar{M}$. Here the warping function is $f = e^{z}$.

BIBLIOGRAPHY

[1] Bejancu A., *CR-submanifolds of a Kaehler manifold,* I, Proc. Amer. Math. Soc. 69(1978), 135-142.

[2] Bejancu A., *CR-submanifolds of a Kaehler manifold,* II, Trans. Amer. Math. Soc., 250(1979), 333-345.

[3] Bejancu A., *Geometry of CR-submanifolds,* D. Riedal. Pub. Co., Boston, (1985).

[4] Bejancu A. and Papaghiuc N., *Semi-invariant submanifolds of a Sasakian manifold,* Anal. Stii. Alc. Univ. Al. I. Cuza. IASI., 27(1981), 163-170.

[5] Bejancu A., *Geometry of CR-submanifolds,* Rudrecht, Boston, Mass: Reidel (1986).

[6] Bishop R. and O'Niell, *Manifold of negative curvature,* Trans. Amer. Math. Soc. 145(1969), 1-49.

[7] Bishop R. and O'Niell, *Manifold of negative curvature,* Trans. Amer. Math. Soc. 145(1969), 356-363.

[8] B Uñal, *Doubly warped products.* PhD. Thesis, University of Missouri-Columbia, (2000).

[9] Blair D.E. and Chen B.Y., *On CR-submanifold of Hermitian manifolds,* Israel. J. Math., 34(1979), 353-363, 2(2000), 125-138.

[10] Blair D.E., *Contact manifolds in Riemannian geometry,* Lecture notes in mathematics Vol. 509, Springer-Verlag, New york (1976).

[11] Bonanzinga Vittoria (Reggio Calabria) and Koji Matsumoto (Yamagata), *Warped product CR-submanifolds locally conformal Kaehler manifold,* Periodica Mathematica Hungarica, Vol. 48(12), 2004, 207-221.

[12] Chen B.Y., *Geometry of submanifolds,* Marcell Dekker. Inc. New York, (1973).

[13] Chen B.Y., *Differential geometry of real submanifolds in a Kaehler manifold,* Monatsh. Math. 91(1981), 257-275.

[14] Chen B.Y., *CR-submanifolds of a Kaehler manifold I,* J. Diff. Geom., 16(1981), 305-323.

[15] Chen B.Y., *Geometry of warped products CR-submanifolds in a Kaehler manifold I,* Monatsh, Math., Vol. 133(2001), 177-195.

[16] Chen B.Y., *Geometry of warped products CR-submanifolds in a Kaehler manifold II,* Monatsh, Math., 134(2001), 103-119.

[17] Chen B.Y., *On isometric minimal immersions from warped products into real space,* Proc. Edinburgn. Math. Soc. 45(2002), 579-587.

[18] Chen B.Y., *CR-submanifold of a Kaehler manifold,* J. Diff. Geometry, 16(1981), 305-322 and 493-509.

[19] Chen B.Y., *Geometry of slant submanifolds,* Katholieke Universitiet Leuven, Leuven, (1990).

[20] Cabrerizo J.L., Carriazo A., Fernandez L.M. and Fernandez M., *Semi slant submanifolds of a Sasakian manifold,* Geom. Dedicata., 78(2)(1999), 183-199.

[21] Gray A., *Nearly Kaehler manifolds,* J. Diff. Geometry, 4(1970), 283-309.

[22] Gray A., *Some examples of almost Hermitian manifold,* Illinois J. Math. 10(1966), 353-366.

[23] Hasegawa I. and Mihai I., *Contact CR-warped product submanifolds in Sasakian manifold,* Geom. Dedicata, 102(2003), 143-150.

[24] Khan K.A. and Khan V.A., *On the integrability of the distribution on a CR-submanifold,* Anal. Stii. Alc. Univ. Al. Cuza. IASI. Spec. Mathematica, (1993).

[25] Kobayshi S. and Nomizu K., *Foundation of differential geometry.* Vol I and II, John Wiley and Sons, New York, (1963), (1969).

[26] Khan V.A., Khan K.A. and Siraj-uddin, *Warped product CR-submanifolds in nearly Kaehler manifolds,* SUT Journal of Mathematics, Vol. 43. 2(2007), 201-213.

[27] Khan K.A., Khan V.A. and Siraj-uddin, *Warped product contact CR-submanifolds of trans-Sasakian manifolds,* Filomat No. 2, 21(2007), 55-62.

[28] Khan K.A., Shahid Ali and Nargis Jamal, *Generic warped product submanifolds in Kaehler manifold,* Faculty of Sciences and Mathematics, University of Nis, Serbia, Filomat No. 1, 22(2008), 139-144.

[29] Khan K.A. and Khan V.A., *Totally umbilical CR-submanifolds of a nearly Kaehler manifold,* Geometriae Dedicata. 50(1994), 47-51.

[30] Kashiwada T., *Some Properties of locally conformal Kaehler manifolds,* Kokkaido Math. J. 8(1979), 191-198.

[31] Khan V.A and Meraj Ali Khan, *Semi slant submanifolds of trans-Sasakian manifolds,* Sarjevo Journal of Mathematics Vol. 2(14)(2006), 83-93.

[32] Ludden G.D., Okumara M. and Yano K., *Totally real submanifolds of complex space manifolds,* Atti. Accad. Naz. Lincci. Rend. Cl. Su. Fis. Mat. Natur., 58(1975), 346-353.

[33] Lotta A., *Slant submanifolds in contact geometry,* Bull. Math. Soc. Roumanic. 39(1996), 183-198.

[34] Matsumoto K., *On CR-submanifolds of locally conformal Kaehler manifolds,* Bull. Yamagata Univ. (N.S.) 11 (1984), 33-38.

[35] Matsumoto K., *On CR-submanifolds of locally conformal Kaehler manifolds, and II,* J. Korean Math. Soc. 21(1984), 49-61 and Tensor (N.S.) 45(1987), 144-150.

[36] Matsumoto K. and Mihai I., *Warped product submanifolds in Sasakian space forms,* SUT J. Math. 38(2002), 135-144.

[37] Mihai A., *Warped product submanifolds in complex space forms,* Preprint.

[38] Munteanu M.I., *Doubly warped product CR-submanifolds in locally conformal Kaehler manifolds,* Monatsh. Math. 150(2007), 333-342.

[39] Mihai I., *Contact CR-warped product submanifolds in Sasakian space forms,* Geom. Dedicata 109(2004), 165-173.

[40] Munteanu M.I., *Warped product contact CR-submanifolds of Sasakian space forms,* Publ. Math. Debrecen 66(2005), 1-2, 75-120.

[41] Munteanu M.I., *A note on doubly warped product contact CR-submanifolds in trans-Sasakian manifolds.* arXiv:DG//0604008 v 1 (2006).

[42] Marero J.C., *The local structure of trans-Sasakian manifolds,* Ann. Mat. Pura Appl. (4) 162(1992), 77-86.

[43] Obiana J.A., *New classes of almost contact metric structures,* Publ. Math. Debrecen, 32(1985), 187-193.

[44] Ogiue K., *Differential geometry of Kaehler manifolds,* Advances in Math. 13(1974), 73-114.

[45] Papaghiuc N., *Semi slant submanifolds of Kaehlerian manifold,* An. Stiint. Univ. Al. I. Cuza. Iasi. Ser. Noua. Mat. 40(1) (1994), 55-61.

[46] Sekigawa K., *Some CR-submanifolds in 6-dimensional sphere,* Tensor N.S. 41(1984), 13-20.

[47] Sahin B., *Non existence of warped product semi-slant submanifolds of Kaehler manifolds,* Geometriac Dedicata. 117(2007) 195-202.

[48] Yano K. and Kon M., *Anti-invariant submanifolds,* Marcel Dekkar, Inc. New York, (1976).